Geometric morphology -
transformations of a conic section archetype

Morten Eide

January 20, 2013

Notes: The page numbers in the document jump from p. 8 to p. 21.

Contents

Part I

Backgroun

d

Classical conic section theory

1. The conic sections and their teachings occupy a very special place in the development of geometry in general. They form p a way the bridge from the very elementary geometry to the higher one. This is already evident in the fact that conic sections are a general concept, appearing in different ways in various forms, and as such can only be perceived as uniform in purely ideal terms. They thus lift themselves out of the purely sensuous constructive geometry towards grasping something ideal in things.

2. There is something peculiar already in the way conic sections were conceived from the very beginning, both in the images we use and in the naming. Cone section means section of a cone, and if we regard the cone as uniform, then the cone sections are created by dividing it up in different . We also see a in the image a direct variant of Plato's cave equation. The conic sections in their multiplicity are shadows of something unified that connects them in an idea. We are also tempted a ag a to Kepler and how he imagines the relationship between God and creation. He sees the sphere as an image of God, while he sees the circle as the creature, which is formed as an intersection between the sphere and a plane. The circle is the original image of the conic sections, so to speak, so that the conic sections are to an even greater extent an image of the manifold created.

3. This diversity is shown to an even greater degree by the fact that the conic section enters into innumerable relationships with other elements, these may be elements that come from outside, such as tangents and chords, or they may be elements that seem to belong to the conic sections, such as axes, diameters and focal points. These relations, properties and laws are *the material* for the morphological investigations that follow, and because they are more or less known today, we will go through the basic relations and properties that apply.

4. Most of these relations as they exist from the classical conic section theory are geometric-metric relations, but as we have seen earlier, we can form incidence relations of these by combinatorial operations. In this review we will not make these transformations, we will see this on a as we encounter a things in the morphology itself.

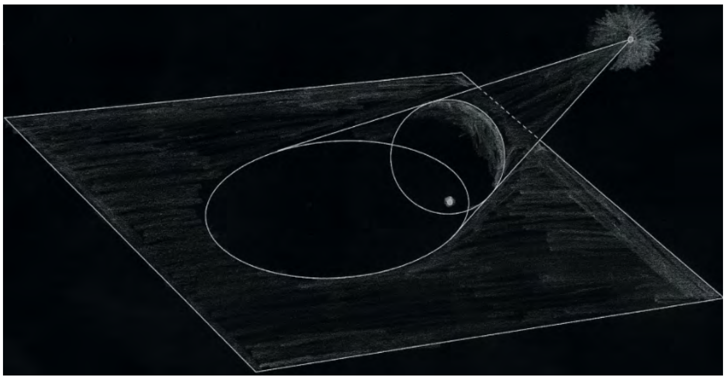

Figure 12: The ellipse as an image of the sphere

The discovery of conic sections

5. Knowledge of the conic section equations can be traced back to the problem of the doubling of the cube, one of those remarkable Delphic problems that contributed so much to the development of mathematics.[2] Here the task is to construct a cubic altar that has twice the volume of another cubic altar. This proved to be problematic with a compass and ruler, because it involves to construct a line segment that is $\sqrt[3]{2}$ larger than another, which has only recently been shown to be impossible.

6. The Greek Menaechmus is creditedwith having discovered that conic sections can be used for this purpose, and he gives two solutions of the problem using them. In one case he uses two parabolas, in the other case a parabola and a hyperbola. He makes use of the numerical relationships associated with these curves, and by solving equations with two unknowns he finds the desired quantity.

7. From quotes, it is also clear that he knows that these curves are formed from sections with a cone. The different cone sections appear as different results of a plane that cuts across a cone. If we cut across so that we get a finite section, we get the ellipse, if we cut so that the plane is parallel to an edge of the cone we get the parabola, and if we cut so that the plane is not parallel, but not finite either, we get the hyperbola. We can see how these appear in Figure 14. Thus, the two aspects of the conic sections emerge from the very beginning, on the one hand the purely geometric, and on the other hand the algebraic conditions associated with them.

---

[2]The other two problems: To divide an angle with a compass and ruler alone; To construct a line segment that is as long as the circumference of a circle n a r we know the diameter.

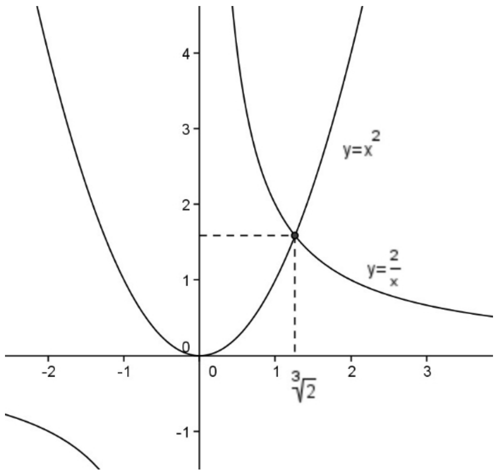

Figure 13: Doubling the cube

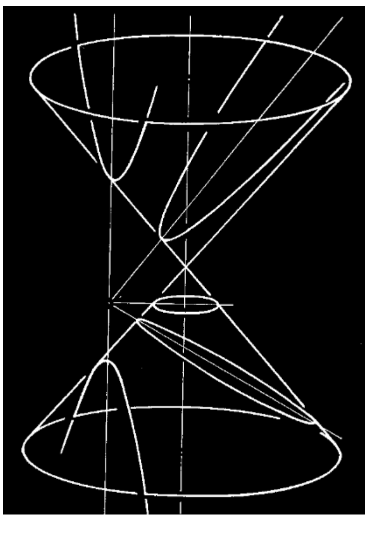

Figure 14: Section of a cone

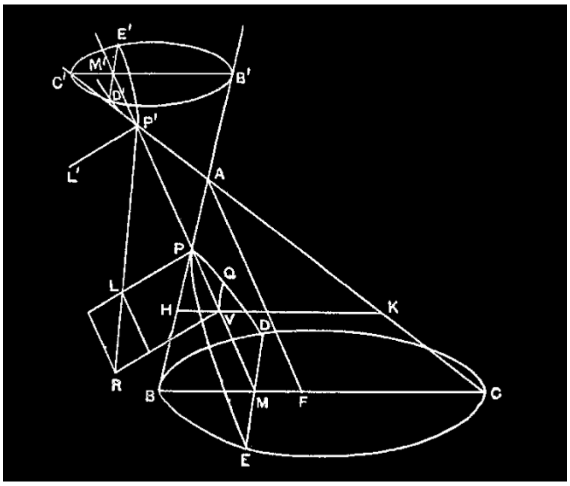

Figure 15: Apollonius surveys

8. In the further study of conic sections, a whole series of mathematicians in ancient times participated; Euclid, Archimedes, Pappos and above all Appolonius. This is not the place to°a give an exact account of how all the results are obtained; we give a sketch and show some features of the m°aten°a work p°a. The main point here is°to°bring out different aspects with a view to°later processing.

9. For°a certain understanding°of this path, we can see how the lining for the parabola emerges from consideration of the knit with a cone.

The equations for the conic sections

10. The conic sections are obtained by°ap°a simplest m°ate cutting across a cone with a plane. Appolonius considered these sections, and he shows how the equations for the different variants are obtained from this. We shall not go°a through this method systematically, but in order for us to have some feeling for the starting point of it all, we will see p°a how the parabolic equation is obtained in the simplest case.

The equation of the parabola is given by $y= a\,x^2$.

Description: We start with a cone which, seen from the side, forms an equilateral triangle, and we begin the section with the unit from the top as we see in° a figure **??**. If°he parabola is created by a plane cutting parallel to a side, this means that the axis of the parabola is always equidistant from the cone.

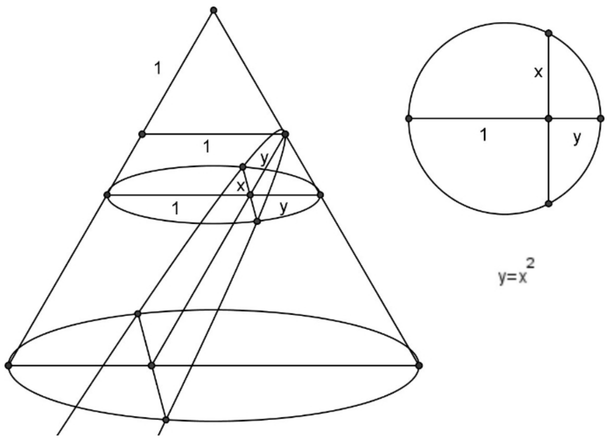

Figure 16: Equation for parabola from section

this page. We then make an intersection normally on the axis of the cone; these intersections become circles, and by following how the circle changes in relation to the conditions on the parabola, the parabola equation emerges. The details of this are left to the reader.

P aligning with ate, we can get the equations for the hyperbola and the ellipse by looking at the other sections. However, we need to see other properties of these equations to obtain the equations for them.

11. The ellipse appears simplest when we look at it as a circle in perspective, and this also gives the equation a simple mate. The simplest representation of this is when a circle s and si are pressed together. All the heights in the circle will then be reduced by the same amount, and based on this, the equation for the ellipse is given by the equation for the circle.

Equation 15. *The equation of an ellipse with center at the origin, and with the coordinate axes as axes is given by*

$$\frac{x^2}{a^2}+\frac{y^2}{b^{(2)}}=1 \qquad (2)$$

Proof: We consider this in relation to the circular equation $x^2 + y^{(2)} = r^2$, and the equations are obtained.

12. An ellipse thus has two symmetry axes and a center. The same applies to the hyperbola, but this does not appear directly as a compressed circle. Another relationship gives an equation for the hyperbola, which initially looks quite different from the ellipse equation. The equation is then given by

$$y = \frac{a}{x}$$

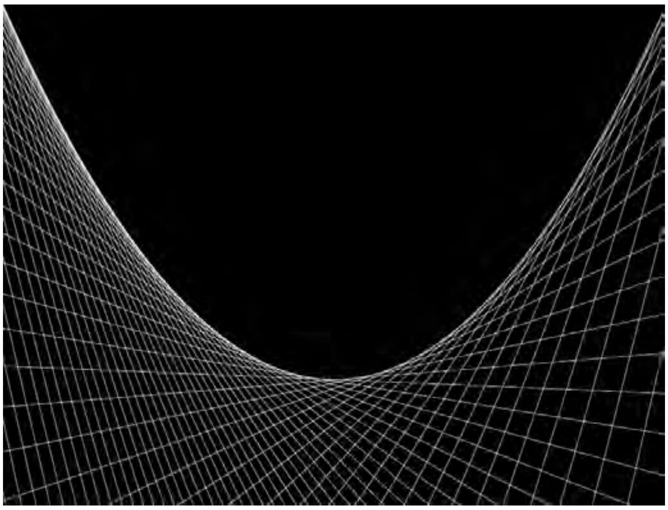

Figure 17: enveloped parabola

. This can also be described as being the curve that gives the same area.

13. The equation above can be transferred to the same type as the ellipse by rotating the coordinate axes. We then replace x and y in the product equation by inserting $x+y$ and $x-y$. This produces the hyperbola equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where *a* is the real axis and *b* is the imaginary axis.

14. The general equation for all conic sections is given by the equation

$$a x^2 + b y^2 + c xy + d x + e y + f = 0$$ We see

nāp a more specific conditions.

Diameters and centers

15. Central elements in conic section theory are *the center* and *diameters* of conic sections, and many properties are associated with these. When the center is given, the diameters of lines through this center are limited by the conic section. If the conic section is given at the periphery, however, the diameter is determined first, and the center is determined as the intersection of the diameters. This is because the diameters are obtained by a simple mate, linked to a central law. If we draw a number of

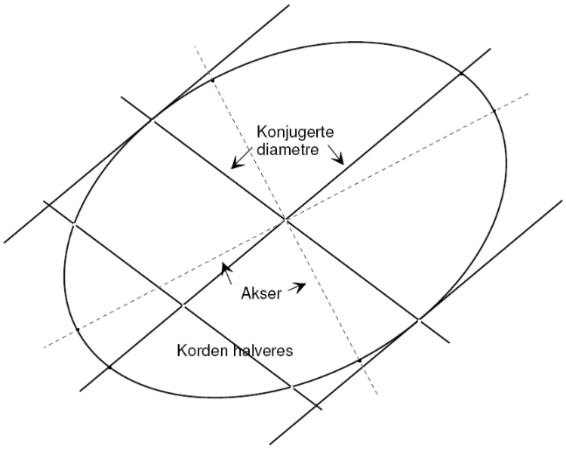

Figure 18: Conjugate diameters and axes

parallel lines across a conic section and halves the resulting chords, turns out that all the bisecting points lie on the same line. This line is a diameter of the conic section.

theorem 16*: Given a conic section, and a number of parallel lines cutting across it. The points of intersection of the chords will then lie on the same line. We call this line a* diameter *of the conic section.*

16. It also turns out that all the diameters formed by this measurement meet at the same point.

Theorem 17*: All the diameters of a conic section meet at the same point, which we call* the center *of the conic section.*

Given a conic section at its periphery, we can use these properties to construct the diameters and center of a conic section.

17. Based on a number of parallel lines, we find °a diameter. Now°a there are lines among the parallel ones that stand out. One of these is the line that also °a g °passes through the center of the conic section, and this is called *the conjugate diameter* of the one found.

18. The other lines that stand out among the parallel ones are *the tangents* to the conic section. A diameter is thus a conjugate diameter of another n °a r it g °a r through the points that parallel tangents form with the conic section.

19. Among the diameters, there are two that stand out, the longest and the shortest diameters. These are called *the axes* of the conic section, and we will here

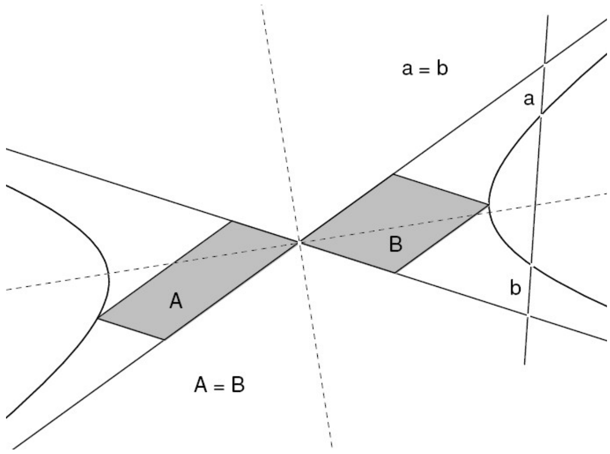

Figure 19: Hyperbola and asymptotes

call these *the long axis* and the *short axis*. We can find these by a sl a circle around the center of the ellipse, and where it intersects this, we halve, and we f ar the axes. The conic sections are symmetrical about the axes.

The hyperbola and its asymptotes

20. Something peculiar is associated with the hyperbola that we do not find for the other conic sections, namely its *asymptotes*. We have already seen these on a, and they exist as two special diameters, and they are also a conjugate to each other.

Theorem 18 *Given two lines. We find points so that the product of the distances from the point to the asymptotes parallel to the other asymptotes is constant. Then the points will form a hyperbola.*

focal point

21. In addition to the center, there are two central points linked to the cone sections, which in many contexts are of great importance. These are the focal points of the conic sections. The phenomena that involve focal points are also a diverse, and they have a different character than those we have seen so far. This distinction can be characterized by the fact that the first phenomena involve parallel lines in one way or another, while the phenomena associated with focal points involve the circle.

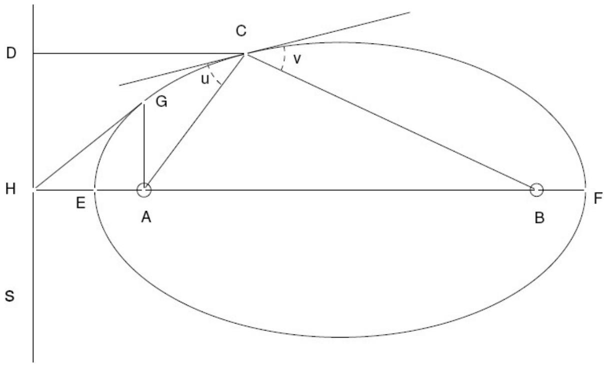

Figure 20: Properties of the ellipse

22. The most immediate way in which the focal points show themselves on a is through the laws of addition.

theorem 19*: **The addition theorem***
Given an ellipse and the two focal points. Then the sum of the distances from a point på on the ellipse to the two focal points is constant, and the sum is equal to the major axis of the ellipse.

23. This relationship can be used to define an ellipse, and in terms of construction we make the ellipse appear by starting with two focal points, and finding points as indicated above.

24. The above definition only applies to the ellis, for the other conic sections, something applies that is actually a modification, but which behaves quite differently. The hy- perbola appears as a constant difference.

Theorem 20 *Given a a hyperbola and the two focal points. Then the difference of the distances from a point på a hyperbola to the two focal points is constant, and the sum is equal to the transverse axis of the hyperbola.*

As for the ellipse, we can make the hyperbola of a appear from this sentence.

Focal point and guidance line

25. N hen it comes to the parabola, something else special occurs. It has only one focal point, and we can regard it as an elongated ellipse where one

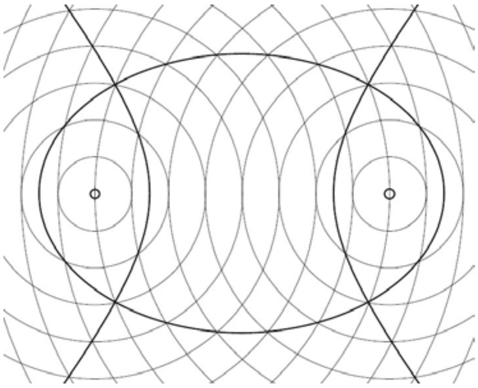

Figure 21: Ellipses and hyperbolas from cosntant sum and difference

focal point has g°att to infinity. This means that sentences relating to two focal points do not apply to the parabola, but we do have a law that we will later show is°directly related to the two above. Our starting point is the focal point and a specific line, the so-called °*control line* of the parabola.

Theorem 21 *Given a parabola, the focal point and the directrix. For all points p °of the parabola, the distance to the directrix will be the same as the distance to the focal point.*

26. We can also °a use this theorem as a definition, and to °a construct a parabola p °a a simple m °ate. It is relatively easy°to find the parabola equation from this context.

27. The guidance line is not only linked to the parabola, it is also °linked to the ellipse and hyperbola. For each of these, a guidance line is linked to each focal point. But in reality, it is not a question of four elements at °a time, but either two focal points, or one focal point and the associated control line. The latter is linked to a central law that applies to all cone intersections.

theorem 22 ***The relationship theorem***
Given a conic section, a focal point and the associated guidance line. From each point p°of the conic section, the ratio between the distances to the focal point and the guidance line is constant.

28. Also°a this statement is used as a definition of°a conic section. From this, conic sections can be constructed, and it is relatively easy°to see the transition to the addition and difference theorems. By°varying the constant, the different conic sections emerge; if ° it is smaller than one we have an ellipse, if it is larger

than one, we have the hyperbola, and if it is equal to one, we have the parabola. We see that the parabola theorem above appears.

Envelope curves

29. So far we have looked at a point occurrences of the conic sections linked to the focal points, but we have also a relationships where tangents are involved. While in the case of points we have to do with lengths, here we to do with the creation of angles. A central theorem, which also a gives all the conic sections, is the following:

Theorem 23: Given a conic section, a focal point to this, and a circle co-centric with this, and diameter equal to the major axis. We draw a line through the focal point, and where this intersects the circle we draw a normal. This will then be tangent to the conic section.

30. By starting with a circle and a point, we can use this law to construct the conic sections. We then draw lines through the point, and where these intersect the circle, we erect normals. All the normals will then enclose a conic section, as shown in Figure 22.

31. By varying the circle f we obtain a number of conic sections. If we start with the point inside the circle, we get the ellipse. We then a fix the point on the periphery of the circle that is closest to the focal point, and let the circle grow. We then get a more and more elongated ellipse, and as the circle becomes a line, the parabola arises. When the circle closes again in the other direction, the hyperbola appears. We also have a dynamic transformation of the conic sections.

32. In the case of the parabola, we can write the theorem specially, because we are no longer dealing with a circle, but with a line.

Theorem 24: Given a parabola, the focal point, the axis, and a normal to the axis where it intersects the parabola. We draw a line through the focal point, and where this intersects the normal, we raise a normal to the line, and this will then be tangent to the parabola.

We will later see that this theorem has significance in and of itself.

33. A key connection, which also has practical significance, is a connection between the point and the tangent perspective.

Theorem 25 Given a conic section, its two focal points, and two lines from these to a point på the periphery. A tangent to the conic section at this point will then halve the angle between the two lines.

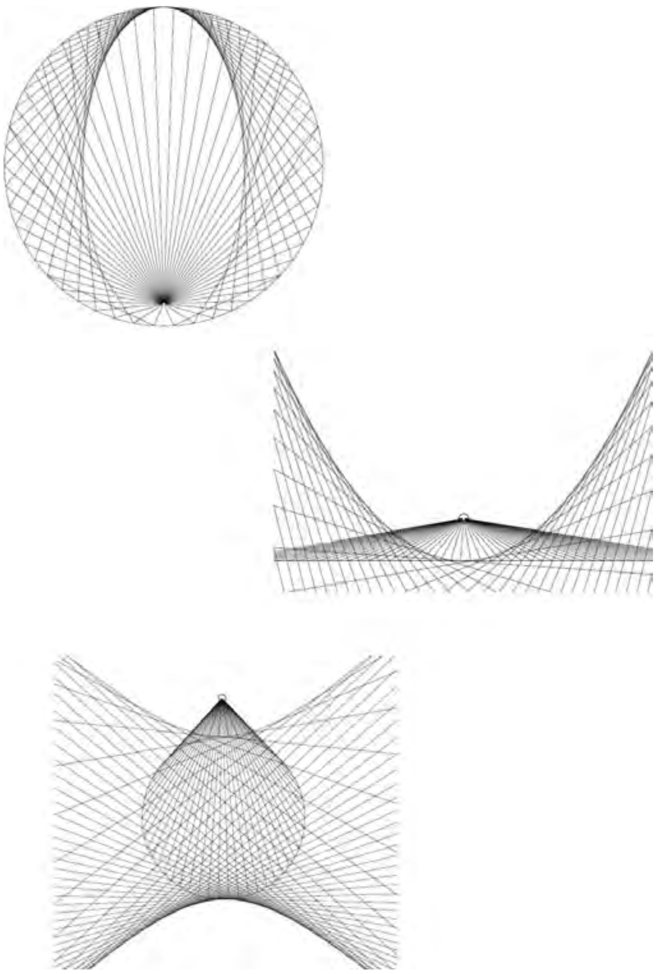

Figure 22: enveloped cone sections

This means that all light emitted from one focal point in an ellipse will be reflected via the periphery to the other focal point.

34. In the case of the parabola, one of the focal points is at infinity, which implies a modification of the above theorem:

Theorem 26*: All lines parallel to the axis of a parabola will be reflected via the periphery to the focal point.*

Parabolic mirrors and satellite dishes use this principle to collect light and radio signals.

Part II

Morphologie

s

Chapter 1 The starting point

In the middle of the 17th century, as mentioned in the introduction, we find the first impulses for what would become modern geometry in the work of Girard Desargues and Blaise Pascal. They developed completely new methods, and these methods are also linked to two central geometric theorems that bear their names. In way, these works carry both aspects of a morphological geoemtry, but the emphasis is on two sides. In the case of Desargues' theorem, we encounter an overarching structure, which also has certain possibilities for transformation in it. Pascal's theorem does not have the same degree of unity in its structure, but it does more possibilities for transformation. In the morphology, we try to unify these sides a.

1.1 Desargues configuration

1. A new element in art at the beginning of the modern era is that visual artists wanted to depict space as real as possible. They therefore sought methods to depict three-dimensional objects exactly as one would see them. Purely concrete depictions were made using a glass plate; from a viewpoint, the various objects were drawn in as they appeared on the plate. This developed into more sophisticated methods where geometric figures were drawn in the plane.

2. Desargues was an architect by profession, and he developed his geometric views from an artistic perspective. He worked with linear structures in perspective, but he also found results for a circle seen in perspective. When considering triangles in perspective, he arrived at a basic theorem: Desargues' theorem.

3. To show Desargues' theorem, we start with a simpler image; a pure enlargement of a triangle. We have given a triangle, and a perspective point inside the triangle. Through the perspective point, we draw str ales through the triangle's

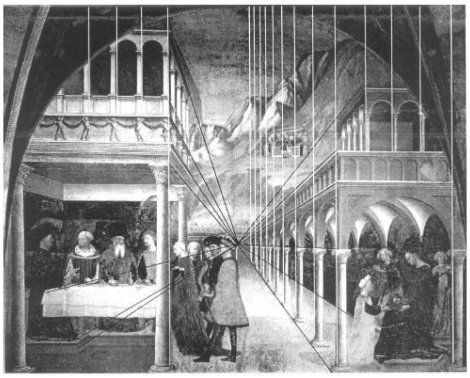

Figure 1.1: Perspective

corners. We then add a parallel to one of the triangle's sides on the outside of the triangle, and where it meets the line alone from the center, we draw new parallels with the other triangle sides. These parallels will then meet on the third line, and we have created a new triangle that is in perspective with the first. This is a pure enlargement, and is the simplest type of perspective (Fig.**??**.A).

4. We see nåpa the image we have f att as a whole, and imagine this entire configuration in perspective. In other words, we imagine how the whole thing would look if this were a figure on the ground viewed from the side. The corners of the triangle will still be connected by three lines through a point. However, the parallel lines will no longer be parallel, they will approach each other at a greater distance, and pa the horizon line they will meet. This applies to all three pairs of parallel lines. The whole picture that emerges is n a Desargues configuration. We can express it in a theorem (Fig. 1.2).

Image 1. ***Desargues theorem***
Given two triangles in perspective. Then matching sides of the triangle will meet in three points that all lie oline. We call this line the Desargues line*, and we can call the perspective point* the Desargues point *in the configuration.*

5. When the three vertices of two triangles lie on the same three lines through a point, we call them *point perspective*. If the three sides of two triangles meet at three points on the same lines, they are called line perspective. Desar- gues can be written briefly based on these definitions:
Desargues' theorem *Nar two triangles are point perspective, then they are also a line perspective.*

6. The configuration can also be regarded as an emergence in space. It can then be seen as a triangular pyramid that is cut by a plane; Desargues

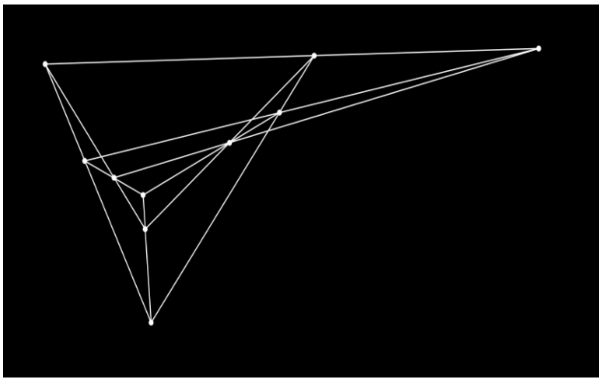

Figure 1.2: Desargues theorem

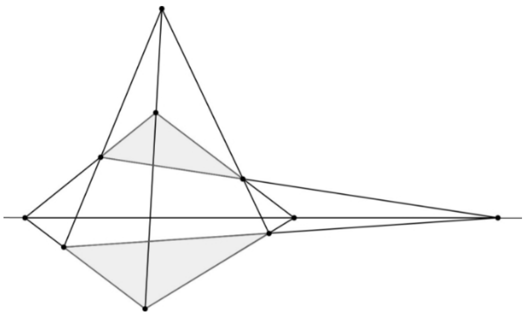

Figure 1.3: Triangle image

The line then becomes the line between this plane and the base plane. Also a as the shadow a triangle, the configuration appears (Fig. 1.3).

7. Desargues's configuration already has the character of a primordial image, because on closer inspection it turns out to be completely symmetrical in a structural sense. In the configuration we find a total of ten lines, and on each of the lines we find three points. There are also ten points, and through each of the points we have three lines. If you look further at the different points and lines, you will find that there is no structural difference between them. This means that all points can be set as perspective points; the choice of point results in two specific triangles, which in turn result in a specific perspective line.

8. If we look at how the configuration appears in space, we become aware of this symmetry. We are dealing with a tetrahedron that is being cut across by a

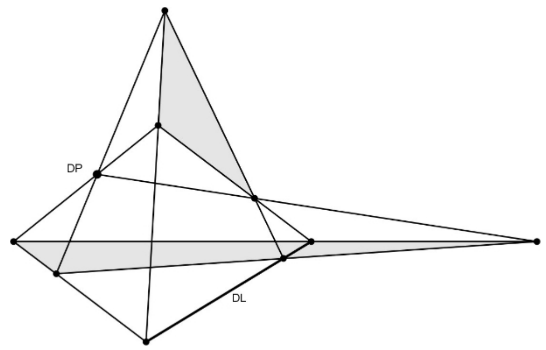

Figure 1.4: Change of perspective point

plane. The structure of the tetrahedron is given by the fourth row of Pascal's arithmetic triangle; it has four planes, six lines and four points, and the numbers are found in the row (1,4,6,4,1). In Desargues' configuration, an additional plane is added, so we are dealing with five planes. These planes form two and two for a total of ten lines, and three and three meet in ten points. We identify these numbers as three numbers in the fifth row of Pascal's arithmetic triangle. This row is given by the numbers (1,5,10,10,5,1). For the sake of completeness, we can also say that the five planes form 5 tetrahedrons.

1.2 Points and line in infinite

9. If we return to the perspective triangles with parallel lines, we don't find the same symmetry as in the full Desargues configuration. Here there are a total of seven points, while there are nine lines. One point, the perspective point, also differs from the other points.

10. It can also happen that only a couple of lines in perspective triangles are parallel. It then turns out that the perspective line becomes parallel to these. The configuration also a changes character here. There are still ten lines, three of which are parallel. However, there are only nine points.

11. In such assessments, the images with parallel lines stand out as special. It is conceivable that Desargues makes such considerations when he introduces the elements at infinity. He says that two lines always meet at a point; n°ar the lines are parallel, they meet at a point at infinity. If two points are at infinity and there is a line through them, then the whole line is at infinity. Thus, the symmetry of the configuration is maintained regardless of the position of the elements, only some elements occasionally lie at infinity.

12. What we are to understand by such a point is to be understood as has been the source much controversy. It is obvious that it is not actually possible to find such a point, and for this reason many were unwilling to accept such a concept. One way to solve the problem was to call the points in infinity ideal points, and thereby distinguish them from those we can construct with. Such a point was then determined by the direction of a line. We shall not deal with this problem in its breadth here, but will return to it when we see a imaginary points, which created even greater controversy.

13. In such a perspective, parallelism becomes something special; the first principle becomes the general Desargues theorem. While we found Desargues' theorem at

To look at the perspective, we put this one first, and then move on to the special images. The connection with two perspective triangles at the beginning is then a consequence of Desargues' theorem: Given that two pairs of matching lines in the configuration are parallel. Then two points will lie at infinity, and the Desargues line passing through them will be *the line at infinity*. The last pair of matching lines that also a meet each other at a this line will thus be parallel.

14. The regularity can be further varied; the original perspective point can be at infinity, or one or more of the corners of the triangles. This gives rise to several sentences that can be justified. An example is the following.

Figure 2. Given two triangles with vertices p a three parallel lines. Then two matching lines in the triangle will meet at three points p of the same line.

15. A new question? What do we really mean by parallel lines in pure geometry. When we do not use metrics, there is actually no answer to this. We can only say that there are lines that meet at a point on a line at infinity, but we can actually place this line wherever we want. Then it will happen that when two pairs of lines in the triangle meet at the arbitrary line at infinity, then the third pair will do so.

16. However, if we have given two actually parallel lines, we can find parallels to this using Desargues' theorem, and if we have given two actually parallel lines, we can find actual parallels to all lines using Desargues' constructions. In this sense, Desargues' theorem acts as a constitutive structure in geometry.

1.3 Desargues theorem as axiom

17. Such considerations mean that today we see p a Desargues theorem as an axiom in projective geometry. That is to say, this is a way of ate saying what we ideally mean by points and straight lines in the plane. N ar we hold

us to the world, then we can use a ruler to draw straight lines and they will then meet as the theorem says. But in doing so we take the concept of line from the external world. By Desargues' theorem, the determination is raised to an ideal viv ° a ; which admittedly corresponds to the external.

18. Defining the concept of a line is easier in space. There, we can ideally consider lines and we realize that they have the following property: *Given two lines that have a common point, and two others that have a common point with both of these. Then the last two must also have a common point.* We realize that lines in space m°a relate to each other like this, but we *put* it at the same time because through this we will characterize a straight line. In the plane, we cannot say anything about lines p°a this m°a, because such a relationship is obvious here. To°a characterize lines in the plane m°a we say that lines are such that can form a Desargues configuration.

19. The fact that we maintain an entire image as s°adan, and from the image find special images by varying the elements in the image, is what we strive for in morphological geometry. Here, Desargues' theorem shows itself as a complete image, which is also° evident in that it is set as an axiom. However, the possibilities of variation are soon exhausted when °we only have points and lines.
å do; a completely different richness is revealed when °we go°into the area of the conic sections°ade.

1.4 Tasks

N°f we don't have a metric, parallelism is justified in relation to given parallelism. If two parallel lines are given, we can find parallels to this one, and if two pairs of parallel lines are given, we can find parallel lines to all lines.

1. *Given two parallel lines, and a point beyond them. Find a parallel to the lines through the point.*

2. *Given two pairs of parallel lines, and a single line that is not parallel to these. Find a parallel to this line.*

3. *Given two pairs of parallel lines, a single line that is not parallel to these and a point that does not lie on either line. Find a parallel to the single line through the point.*

Another aspect of Desargues' theorem is that each point can be a perspective point. Each new point we choose results in two new triangles, and a new Desargues line.

4. *Given a Descartes configuration. Starting from another point as the perspective point, find the perspective triangles, and the Desargues line. This can be repeated with other points.*

5. *Given a Desargues point, a Desargues line, and two matching lines in the triangles. In addition, a corner* p° *of one triangle is given. Find the corresponding point on other triangle.*

Chapter 2

Pole and

polar

Desargues also applied his principles °ato°a conic sections, by°a considering them as circles in perspective. Several of the sentences known from Greek times° find their elementary explanation by this way of°ate °a understanding°a the phenomena p ° a. With the introduction of the terms pole and polar, there is a general realization of relationships around centers and diameters. Here is already in germ what would become a main method in the projective geoemtry;°a understand°a more complicated relationships by°a projecting to and from simpler and immeasurable conditions.

20. From Greek geometry, we know a special relationship related to the dimensions and centers of the ellipse.

Figure 3*: Given a* diameter *through* the center *of the ellipse. This intersects the jaw section at two points, and tangents at these points will then be parallel. The line through the center parallel to the tangents is called* the conjugate di- ameter *of the first, and tangents where this intersects will be parallel to the first diameter.*

21. The above theorem is obvious °when it comes to circles. If we take a picture of a circle as our starting point and put it into perspective, the elements change position, but certain conditions apply. The parallel lines meet three by three at °two points on °the peripheral line. The resulting image can be expressed:

Figure 4*: From a point outside a conic section, we draw tangents to it and a line through the tangent points. We draw a new line through the original point, and where this meets the conic section we add tangents. These meet at°a line between the tangent points.*

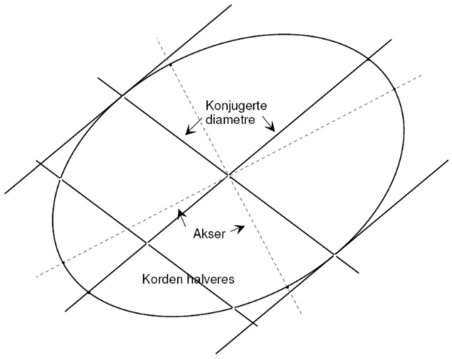

Figure 2.1: Conjugate diameters and axes

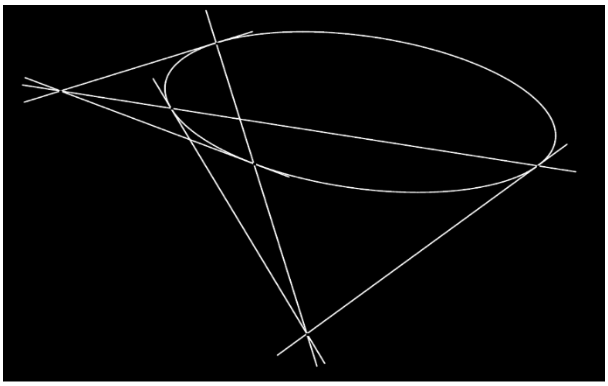

Figure 2.2: The Hire theorem

The theorem is called La Hire's theorem.

22. Here we have arrived at a projective image for conic sections. None of the lines are parallel anymore, and the theorem indicates a certain relationship that applies to conic sections in general, although it is currently only clear that it applies to ellipses. The above theorem has a certain transformability in it, and as we shall see, it can be generalized to other more general theorems.

23. In the resulting image, we have two points outside the conic section, and each of the points is connected to a specific line formed by the tangent points given by the point. We denote n a such a point as *the pole* of the line through the tangent points, and the line then becomes *the polar* of this point.

24. We can shorten the above theorem by using the terms pole and polar: *Given a pole and polar. We add a new pole to the polar, then the polar of this will go through the pole.*

25. If we have given a polar outside an ellipse, the pole will lie inside it. From points p a the line, we find polars that intersect the ellipse, and these all meet at the pole of the line. This is related to the center and the horizon line in the image above.

26. Conversely, we find the poles of points that lie inside the conic section. We then draw two lines through the point, find the poles of these lines, and the line through the poles is the polar of the original point. Thus we can find the poles of all lines, and the poles of all points with respect to an ellipse.

2.1 Center and diameters

27. From the terms pole and polar, we can give projective definitions of the diameter and center of an ellipse. If we let the pole ga to infinity, the polar becomes a diameter.

Definition 1. *A diameter of a conic section is the polar of a point pa line at infinity.*

28. If we instead let the pole ga to infinity, the pole becomes the center:

Definition 2. *The center of a conic section is the pole of the line at infinity with respect to the conic section.*

From this we can conclude that all diameters go through the center, because all the poles of the diameters lie on the line at infinity.

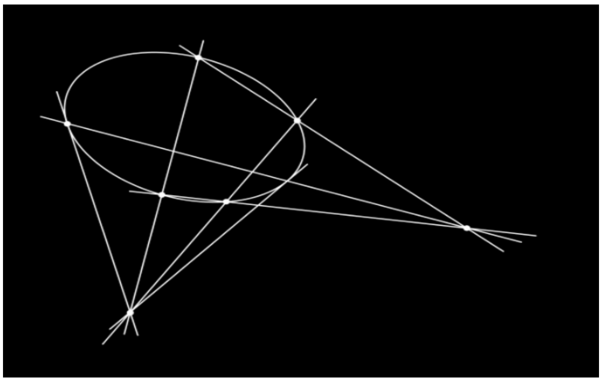

Figure 2.3: Expanded La Hire theorem

2.2 Extended La Hires theorem

29. In the images above, we repeatedly draw parallels from a point to the jaw section. From La Hire's theorem, however, we cannot construct the tangents exactly. However, a certain extension of the theorem makes this possible, and we arrive at this by °a first looking at °a a circular image:

Figure 5: Given a circle, two parallel tangents to it, and two lines parallel to them. Lines through the points of intersection will meet at the diameter between the tangent points.

We realize this immediately by the symmetry.

30. N °a r the image above is in perspective upst °ar the theorem:

Figure 6. Given a conic section, a point outside the conic section, the two tangents from the point to the conic section, and the polar through the tangent points. We draw two new lines from the point. Where these meet the conic section, we draw common lines, and these will meet at the pole.

We call the theorem the extended La Hire theorem.

31. Extended La Hire's theorem g °changes to La Hire's theorem n °when the two lines approach each other and finally coincide. Then the lines through the points will become tangents to the conic section in the common point, and La Hire's theorem appears. We will take a closer look at this type of transition later.

32. By extending La Hire's theorem, we can find the pole of a conic section n ° a r the pole is given. We can do this immediately because two lines can form two pairs of common lines that meet at the pole, and this can then be drawn. We could

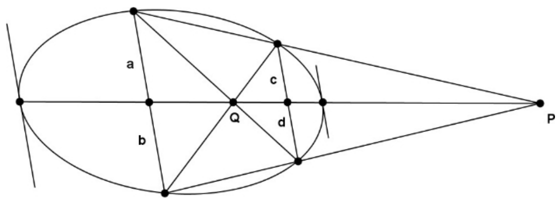

Figure 2.4: bisecting of chords

We can also draw more lines across the conic section and find more points on the pole. This also means that we can find the tangents from a point to a conic section n when this is given; we first find the polar, and from this the tangent points.

33. If the pole is inside the conic section, we can also use the theorem, although we cannot draw tangents in this case. Just as above, we draw lines through the pole, where these meet the conic section, we draw lines, and these meet at the pole.

34. By extended La Hire, we can realize a classic phrase that has multiple uses. This is the theorem about shared chords:

Metric law 1. *Given a diameter of a conic section, and a chord parallel to the conjugate diameter. The chord is then divided into two equal line segments of the diameter.*

To justify the theorem, we leave the pole in extended La Hire at infinity. The tangents from the point will be parallel, and we find the pole through the tan miter points. The other lines will be parallel to the conjugate diameter. We can see from figure 2.4 that there are equilateral triangles seen from b ade P and Q that give:

$$\frac{a}{b} = \frac{c}{d} \; ; \; \frac{a}{b} = \frac{d}{c}$$

$$\Rightarrow \frac{a}{b}\frac{a}{b} = \frac{c}{d}\frac{d}{c} = 1$$

$$a = \pm b$$

Chapter 3 Pascal's hexagram

Although Desargues introduces projective geometry, it can be said that Pascal is the one who really introduces the concept of metamorphosis. By his discovery of the mysterious conic section theorem, which Pascal called "Hexagram Mys- ticum", and by his treatment of it, he lays the seed for what becomes morphological geometry. The morphological theme of Desargues' theorem is the consideration of points and the line at infinity. In Pascal's theorem, two other motifs of metamorphosis emerge: one is the coincidence of points, and the other is that the conic section itself is transformed and becomes a pair of lines. These different motifs mean that the possibilities for transforming the theorem are much greater; it is said that Pascal himself found 400 corollaries to it.

At the various transitions that are made, special conditions occur, which mean that various known aspects of the conic section emerge. Of particular importance is the fact that the equations for the conic sections appear, which means that we can use Pascal's theorem as a starting point for conic section theory.

3.1 Pascal's Hexagram Mysticum

Just as Desargues' theorem appears as triangles in perspective, Pascal's theorem can, from one point of view, be seen as a circular structure with parallel lines seen in perspective.

We start with a circle and draw two parallel lines across it. For the sake of clarity, we place them not too far apart and let the chords span about 120 degrees. The parallel lines will na intersect two arcs pa the circle. From two of the points of intersection with the circle; from two points next to each other, we draw two new parallels, and these intersect a new arc of the circle that is the same size as the first two. If we draw na lines between these points and the original one, then a these lines will also be

Figure 3.1: Pascals

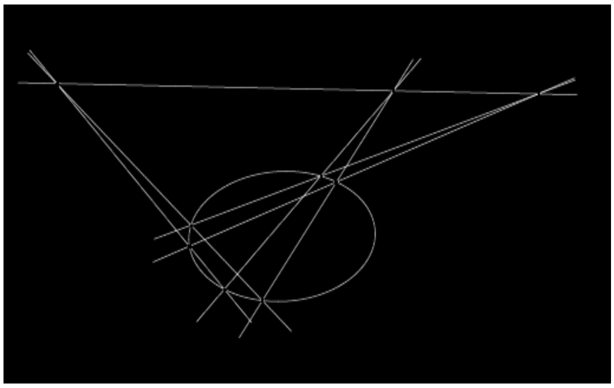

Figure 3.2: Pascal's theorem

parallel because they cut across the same arc. If we follow the lines, we will see that they form a hexagon (Fig. 3.2.A).

Figure 7. *If two pairs of lines in a hexagon inscribed in a circle are parallel, then the third pair will also be parallel.*

We see nåpa the circular image we have in mind in perspective as we did with the triangle image. Two things will happen: the circle will become an ellipse, and the three pairs of parallel lines will meet on the common horizon line. We thus have the theorem (Fig. 3.2.B)

Figure 8 ***Pasclas theorem*** *Given a hexagon inscribed in a conic section. Then opposite sides of the hexagon will meet at three points that all lie on the same line.*

We have thus arrived at a variant of Pascal's theorem. The introduction here is not a definitive proof; what is presented is intended to make the connection probable. We thus assume Pascal's theorem, and see what is consequences of it. The fact that it leads to certain things has a reciprocal effect; if more known things can be derived, then we become familiar with the theorem and can it further in the morphological considerations.

The first thing we can realize is that we can place the hexagon however we want on the conic section. It is not a question of a hexagon as a view, but as a structure, where we stretch lines between six completely arbitrary points on the periphery. It will always happen, however, that three pairs of opposing sides, in a structural sense, meet at three points on the same line.

Here, a combinatorial question can also be raised. Given six points on a per- ifer, as many as 64 different hexagons can be formed between the points, and this gives rise to 64 pascal lines. It can also be investigated how the pascal lines are located; you will always find points where a pascal line meets the intersection of other pascal lines. However, these questions alone will not be explored here.

Although we can lay out the hexagon any way we like, there are some configurations that turn out to be more important than others because they are the starting point for the various morphological movements. In particular, there are two configurations of Pascal's theorem that come to mind. In one configuration, which we here call Pascal1, we have an ordinary hexagon p a an ellipse, but so that the main weight of the points lies towards an edge. When we draw opposite sides of this, they will meet in three points located on the Pascal line outside the ellipse.

In the second image, which we call Pascal2, we let the lines in the hexagon cross each other. The points in the hexagon are placed in the order 1,5,3,6,2,4 around the ellipse, and when we draw lines in the natural order 1,2,3,4,5,6, points of intersection are formed between opposite sides inside the ellipse, and the Pascal line g a r s over the ellipse. (Fig.3.3)

Pascal's theorem applies in the same way to all conic sections, whether circle, ellipse, parabola or hyperbola. But certain things occur for each of the conic sections, and this is what makes them stand out as distinct.

Before we look at the morphological themes associated with the properties of the particular conic sections, we look at a transformation of the conic section itself; which will not play a further role in the consideration of Pascal's theorem, but which will later prove to be of the greatest importance. It is a matter here of the conic section degenerating as we say, and becoming two lines. This will result in a configuration of only points and lines, and the resulting law is called Pappo's theorem. The starting point is Pascal2, and the ellipse is elongated so that it becomes two lines.

Figure 9: *Given two lines, and three points on each of the lines. We form intersections of lines between two and two points på each line, and the intersection points are located*

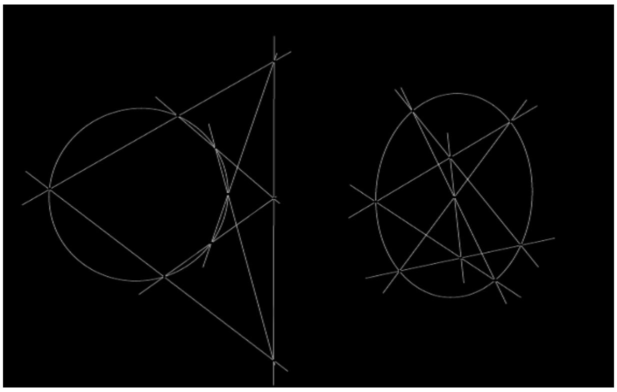

Figure 3.3: Variants of Pascal's theorem

p has the same line.

Pappo's theorem was already known in Greek times, and is one of the fit insider theorems from that time. Like Desargues, it is a completely symmetrical theorem, consisting of 9 points and 9 lines; there are three points on each line, and three lines through each point. Also a Pappo's theorem plays a role in the axiomatization of projective geometry.

As I said, we will see later what significance this transition has. When several conic sections are included in the images, this transition will occur in many ways. We then move on to the conic section morphology.

We are going to look at two main topics; how constructions arise that allow us to define the different conic sections separately, and how the equations for their creation arise. In order to achieve this, we will make movements with the configuration so that special variants emerge. Each variant is suitable for its formal. There are two different movements that change the character Pascal's configuration so that new ones arise. One movement is that lines become parallel, as in Desargues' configuration. The other is that points on the periphery coincide. This causes tangents to arise, and special configurations appear.

The first movement we make is to let two lines in the structure be parallel. These will then meet at infinity. The Pascal line that goes through this point will then also be a parallel to these lines, so that we f have the theorem.

Figure 10. *Given a hexagon inscribed in a conic section, where two opposite sides are parallel. Then the Pascal line of will be parallel to these lines.*

When two pairs of lines become parallel, we will hae two points in finity. The Pascal line that g ar through these points will thus become the line at infinity. The third

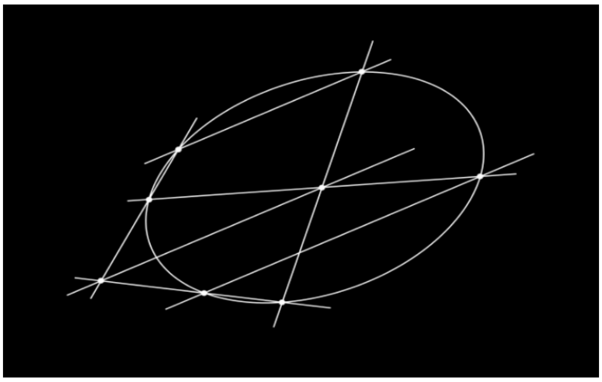

Figure 3.4: Parallel lines

The pair of opposing sides must also meet on a the Pascal line, and must a consequently also be parallel. From this we get the connection.

Figure 11. *If two pairs of opposite sides of a hexagon inscribed in a conic section are parallel, then the third pair of opposite sides will also be parallel.*

We see that these sentences are analogous to two of the sentences related to Desargues configuration.

3.2 Coinciding points

By moving the points on the conic section periphery, it can happen that two points with a common chord merge into one point. It is then no longer possible to find the connecting line between them. However, if we follow what happens as the point approaches, we will see that the line between the points eventually becomes a tangent in the double point. This is a purely geometric process that corresponds to a differential process in the analysis. When we do not use analytical methods, we must put this as we find it intuitively. We cannot prove this in the usual sense, but set it axiomatically.

Axiom 1. *If two points in a Pascal configuration coincide, the line between them will be the tangent in the double point.*

Whenever we have two points coinciding we use this axiom. There are several ways in which one or more pairs of points coincide, and each of these ways gives rise to particular theorems with particular properties. Each of the resulting variants will eventually be applied to its particular formal, and we will go a systematically through these.

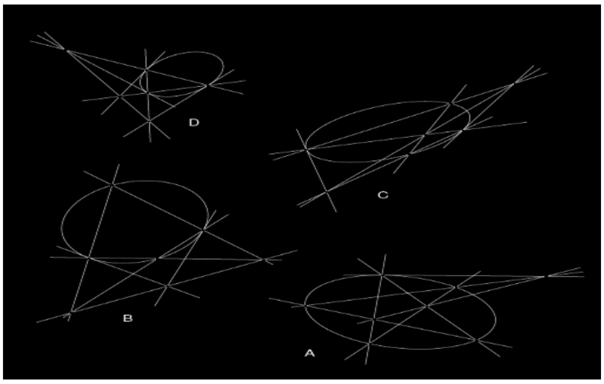

Figure 3.5: Coinciding points

Pentagon configuration

If a couple of points coincide, we will have a pentagon left, with a tangent in one of the points. (Fig.3.5.A)

Figure 12. *Given a pentagon inscribed in a conic section. Then we want a tangent at a point, a line in the pentagon, and a line through two intersections between the other four lines, go through the same point.*

This theorem can be used constructively to find a tangent at a point where we have given a conic section. We then find the Pascal line at two pairs of opposite sides, and where this meets the opposite side of the tangent, we draw the tangent to the tangent point.

Two pairs of points in the Pascal configuration can coincide in two different ways. It may be that there are two pairs of neighboring points that coincide, or it may be that the points form opposing sides. In both cases, squares are formed, but the character of the configurations is different. The theorem that arises when two pairs of opposing points coincide is called Mclaren's theorem (Fig. 3.5.B).

Image 13. ***McLaren's theorem***

Given a square inscribed in a conic section. Then opposite sides, and opposite tangents will meet at points that all lie on the same line.

This theorem has its own name after the English mathematician McLaren, and is then called McLaren's theorem.

In order to have a name for the second theorem that arises when two pairs of points coincide, we call it McLaren's second theorem. (Fig.3.5.C)

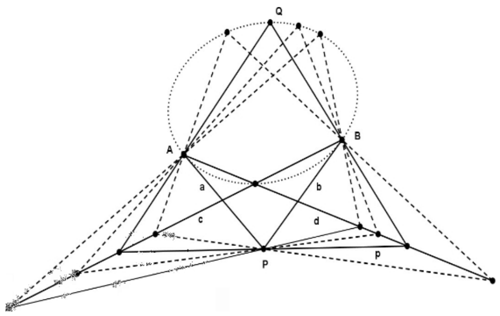

Figure 3.6: emergence of conic section

Figure 14. ***McLaren's second*** *theoremGiven a square inscribed in a spline, and tangents at two neighboring points in the square. Sides of the square and tangents meet at two points, the line between the tangent points and the opposite side of a point, and these points lie on the same line.*

The last variant of configurations with coinciding points far we nar three pairs of points coincide. There is only one variant here (Fig. 3.5.D).

Figure 15: *Given a triangle inscribed in a conic section. Then the sides of the triangle will meet the tangents in opposite corners at three points that lie on the same line.*

3.3 Construction of the different cone sections

In addition to giving properties to the conic sections, Pascal's theorem also determines these implicitly. If we have given five points on a conic section, we can, by applying the theorem, find as many new points as we need. We can realize this if we start from a given conic section with a Pascal con- figuration. If we allow any of the points to move to the periphery, then two of the lines will follow. As a result, the intersection points with the opposing sides will , and the Pascal line will revolve around the third point on the Pascal line. If we turn this around and allow the Pascal line to rotate, the process will be reversed, the point p on the periphery will then move and it will be able to draw the conic section.

This construction forms the basis for all constructions of conic sections based on Pascal's theorem. The Pascal line turns, two lines follow it, and their intersection draws the conic section.

By°placing the starting points in°the right places, the three different conic sections - ellipse, parabola and hyperbola - will emerge, and we will see that these differ in their relationship to the line at infinity. For construction, however, we do not use the general variant of Pascal's theorem, but McLaren's theorem, because the constructions here are simpler, and because the tangents eventually make themselves felt.

We take as our starting point the McLaren theorem that emerges from Pascal1. (Fig.3.5.B) By°holding the top point of°the conic section fixed while turning the Pascal line, the bottom point will move along the conic section, thus describing it. The actual construction method°can also°be described explicitly.

We start with two lines a and b that form a V, and these meet at P. We set a point A p°a on one line, and a point B p°a on the other line. Through points A and B we draw lines c and d, which intersect below A and B. We draw s°a line p through P approximately horizontally, and this line meets c and d at their respective points. From these points, we draw lines through A and B, and these meet at Q. N°ar s°a the line p revolves around P, Q will describe a conic section.

We could also°a take lines through Q as our starting point: lines through P intersect these at two points, and lines through these and A and B also°a form points on°a the underside of the conic section.

The resulting conic section is an ellipse. By°moving Q outwards, and finally to infinity so that the lines through Q become parallel, a parabola construction is produced.

If n°a the lines c and d in the construction image turn even slightly, then the point Q will not be formed above the points A and B, but below. If we do this so that Q remains below P, the construction will produce a hyperbola. By°a turning the line p we will see that points Q in this case g°ar towards infinity, for s a°a lie in infinity n°ar lines become parallel. S°a Q will reappear on°a the underside of P where it also°a describes an arc, before it again disappears to infinity. S appears s° a again from the other side, and the arc eventually closes.

The hyperbola will p°a this m°aten appear by a continuous movement, and the two parts of this naturally belong together in the light of points at infinity. We can say that a hyperbola is a conic section that intersects the line at infinity. In this view, the parabola is a conic section that is tangent to the line at infinity, while the ellipse does not touch the line at infinity. Based on this, we define the different conic sections.

Definition 3. *A hyperbola is a conic section that cuts across the line at infinity, a parabola is a conic section that is tangent to the line at infinity, and an ellipse is a conic section that neither cuts nor is tangent to the line at infinity.*

3.4 Hyperbelens asymptotes

We can make a special change to the above construction for the hyperbola. We let the two points A and B på the lines a and b gå to each other. The two lines c and d that går through these points will thus be parallel to a and b, and we let them meet at the point Q above P. We draw the line p through P, and this meets c and d at points C and D. The lines from these points to B and A will also å necessarily be parallel to the lines a and b, and they will meet at the point S. Når p dreier, S will describe a hyperbola. We emphasize nå that all the conic sections produced by this construction are tangent to lines a and b at points A and B. In the construction above, the points of tangency are therefore at infinity, and the lines a and b will therefore be tangential to the hyperbola at infinity. These lines are nå the *asymptotes* of the hyperbola. In the morphological geometry, we thus define the asymptotes of å to be the lines tangent to the hyperbola where it intersects the line at infinity.

Definition 4. *The tangents of the hyperbola where it intersects the line in infinity are the asymptotes of the hyperbola.*

A construction for the hyperbola based on the asymptotes can thus be expressed explicitly: Given two lines (a and b) through P, and a point (Q) between the lines. Parallels (c and d) with a and b through Q intersect the line p (which går through P) in C and D, and parallels through these points intersect each other in S. Når p dreier vil S tegne en hyperbel med a og b som asymptoter.

3.5 Metric conditions

If we really have a hyperbola in the above construction, we can realize when we study the metric relationships that arise. When one or more lines in the configuration become parallel, similar triangles of different kinds may arise, and from consideration of the relationships inherent in this, we can infer relationships that apply to the conic sections. For the hyperbola above, we find a known relationship by considering the static variant of the construction above.

Figure 16: *Given a hyperbola and its asymptotes, and two points på its periferi på either side of the center. Parallels with the asymptotes are drawn through the points, and the line through their intersection points will then also gå pass through the center.*

If we nå see på the pure point line structure that is formed, i.e. we å disregard the hyperbola, then we can recognize it as the point line structure of a well-known metric relation. This says:

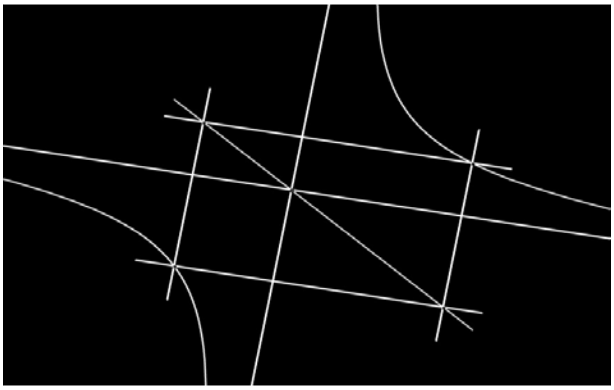

Figure 3.7: Tangents are asymptotes

Metric law 2. ***Gnome composition***
Given a rectangle, a diagonal in it, and two lines parallel to the sides that meet on° the diagonal. This creates two rectangles, one on each side of the diagonal, and these are equal in size.

This immediately gives a metric relationship for the hyperbola. The two areas are formed from the points on the periphery, which means that all areas formed in this way are of equal size.

Metric law 3. *Given a hyperbola and its asymptotes. From a point på per- ifer, we add parallels to the asymptotes, and these together with the asymptotes form a parallelogram. When °the point moves, the area of the parallelogram will be unchanged.*

McLaren2 f°is a definite shape n°is the two points of tangency g°is to infinity, and the tangents here become asymptotes. (Fig.3.8)

Figure 17: *Given a hyperbola and its asymptotes , and two points on the periphery. A line through these points, each parallel to its own asymptote, forms new trap points with the asymptotes. The line through these is then parallel to the line through the points påt the periphery.*

This configuration gives rise to a metric context.

Metric law 4. *Given a hyperbola, its asymptotes, and a line that cuts across the hyperbola. Between the line*

P°due to the parallelism, we see that the line through the points p°a the periphery forms line segments equal length; *a*= *b*= *c* (Fig.3.8).

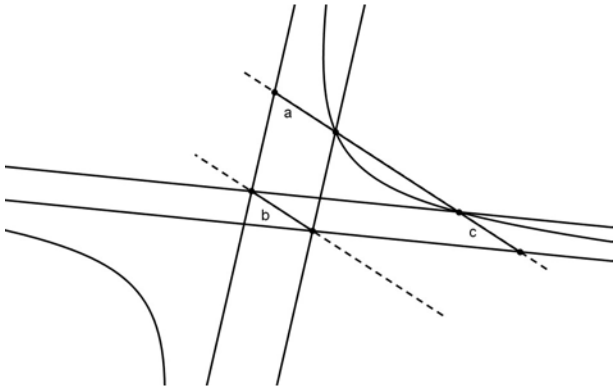

Figure 3.8: Equal length law

Legality can be used to construct the hyperbola. Given two lines and a point, draw lines from the point and subtract the distance from the point to one line from the other line. These points will then form a hyperbola with the lines as asymptotes.

McLaren2 can also be used to find a metric relationship on elements. We then let the two tangents be parallel, and let the line through the two points on the periphery be parallel to the line through the tangent points. Then we have the following:

Figure 18: *Given an ellipse, two tangents to it, and two points on the periphery such that the line through them is parallel to the line through the tangent points. We draw lines between the tangent points and the points on the periphery, and where these meet the tangent points are formed, and lines through these are then parallel to the line through the tangent points.*

By studying the configurations here, we see that the distances from the points på the periphery to the tangents are the same length.

Metric law 5. *Given an ellipse, two tangents to it, and the line (l) through the tangent points. A line parallel to l forms line segments of equal length between the ellipse and the tangents.*

3.6 Equations for the conic sections

The knowledge of the conic section equations is traced back to the problem of the doubling of the cube, one of these remarkable Delphic problems that helped

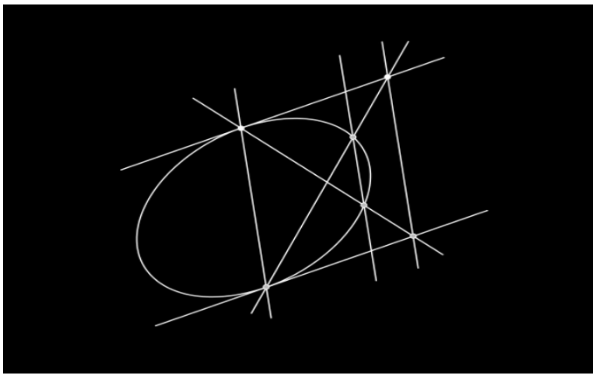

Figure 3.9: Parallelism ellipse

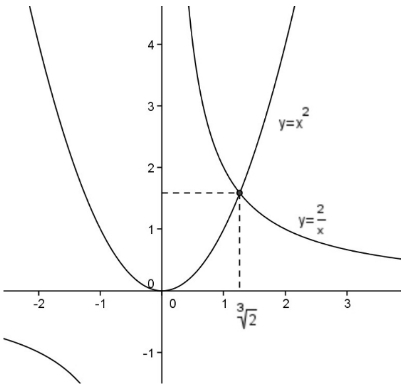

Figure 3.10: Doubling of the cube

s a lot to the development of mathematics. [1] Here the task is to construct a cubic altar that has twice the volume of another cubic altar. This proved to be problematic with a compass and ruler, because it involves to construct a line segment that is $\sqrt[3]{2}$ larger than another, which has only recently been shown to be impossible.

The Greek Menaechmus is with having discovered that conic sections can be used for this purpose, and he gives two solutions of the problem using them. In one case he uses two parabolas, in the other case a parabola and a hyperbola. He makes use of the numerical relationships associated with these curves, and by solving equations with two unknowns he finds the desired quantity.

Ne have already seen how metric relationships emerge from Pas-

[1]The other two problems: To divide an angle with a compass and ruler alone; To construct a line segment that is as long as the circumference of a circle n a r we know the diameter.

cals theorem, s°a, finding°equations becomes an extension of this, because equations°a can also be regarded as metric relationships. It is therefore possible to
Finding the equations of the various conic sections from similar considerations we made when°we found metric ratios. We are not going to find the very general equations; that can be done by transformation within analytical geometry. Here we will see that the basic equations appear. For the parabola, we will see that the equation $y=x^2$ appears, and for the ellipse and hyperbola
we will arrive at the symmetrical equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

We immediately see how an equation for the hyperbola n°ar arises from the area theorem n°ar the x-axis and y-axis are asymptotes. We f°ar in this case $xy=k$ or $y=\frac{k}{x}$.

To°a obtain°a the other equations, we start a particular variant of McLaren1, which we can also°a use to°a construct the conic sections. We imagine a conic section and add parallel tangents to it. The Pascal line will thus become parallel to these lines, and the theorem changes character.

Figure 19: *Given a conic section and two parallel tangents to it. From two points on° the periphery of the conic section, we draw lines through the tangent points, and these will intersect at two more points. The line through these points is then parallel to the tangents.*

Parabolas

This configuration f°ar a special expression for the part of the parabola. We imagine an ellipse with two tangents, but let s°a this become infinitely elongated so that it becomes a parabola. One of the tangents will be at rest, while the other becomes the line at infinity. In this way we f°ar a modification of the theorem above.

Figure 20: *Given a parabola, a tangent to it, and two points p°at the periphery. From the points p°a the periphery, we draw lines to the tangent point, and lines parallel to the parabola. These meet at two points, and the line through these points is parallel to the tangent.*

From this theorem, we can n°a to the elementary parabola equation quite directly. We then let the tangent to the parabola be the x-axis, and let one point p°a the periphery be (1, 1). (Fig.4.9) By°a looking at the equilateral triangles, we see how the equation is easily obtained. We have

$$\frac{y}{x}=\frac{x}{1} \Rightarrow y=x^2$$

The equation can be generalized by°a choosing other values for the fixed point.

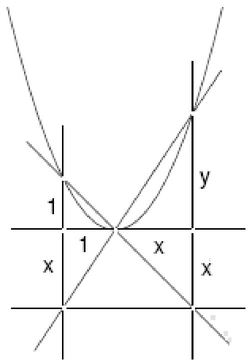

Figure 3.11: The equation of the parabola

Ellipses

We apply to show the equation for the ellipse.

Metric law 6. *The equation for a central ellipse is given by:*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

We let the ellipse have a center at the origin with the longest axis på the x-axis. We place the parallel tangents where the ellipse intersects the x-axis, and we choose one point på the periphery where it intersects the positive y-axis. The other point is freely chosen and has coordinates (x,y). Lines from the points of intersection with the x-axis through these points will intersect at two more points P and Q, and the line through these is then parallel to the y-axis. At
By comparing the shaped triangles that are formed, we will be able to set up the following expressions.

$$1)\ \frac{PR}{AR}=\frac{b}{a}\quad 2)\ \frac{QR}{BR}=\frac{b}{a}\quad 3)\ \frac{QR}{AR}=\frac{y}{a+x}\quad 4)\ \frac{PR}{BR}=\frac{y}{a-x}$$

We can notice that the product of the right's 1) and 2) is equal to the product of the left's 3) and 4). Thus, å the left's products are also equal, which leads to

$$\frac{b}{a}\frac{b}{a}=\frac{y}{a+x}\frac{y}{a-x}$$

$$\Rightarrow\quad \frac{b^2}{a^2}=\frac{y^2}{a^2+x^2}$$

$$\Rightarrow\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

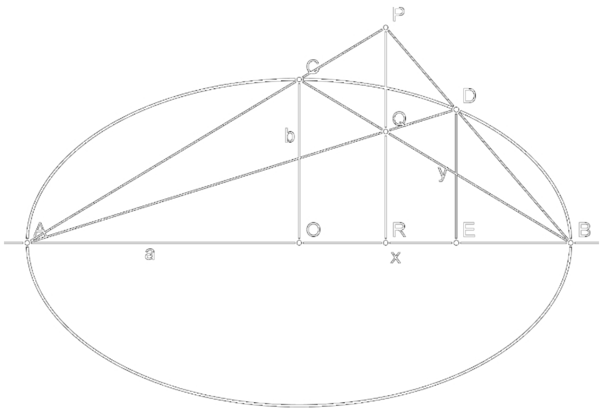

Figure 3.12: The ellipse equation

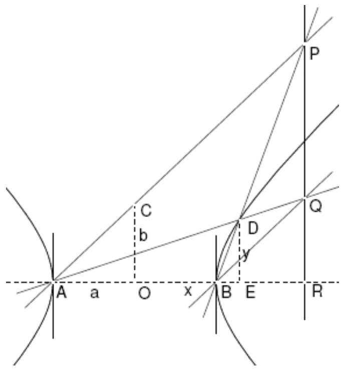

Figure 3.13: The equation of the hyperbola

Hyperbolic

As mentioned above, it is not easy to find the equation for the hyperbola n when it is symmetrical about the axes. Here, however, we are also using the asymptotes; we choose one point at infinity p of the hyperbola. This forms lines parallel to the asymptotes, and equal triangles are also formed here. In the same way as for the ellipse, we find various similar relationships.

$$1)\ \frac{PR}{AR}=\frac{b}{a} \quad 2)\ \frac{QR}{BR}=\frac{b}{a} \quad 3)\ \frac{QR}{AR}=\frac{y}{a+x} \quad 4)\ \frac{PR}{BR}=\frac{y}{x-a}$$

The only difference from the ellipse ratios is that the denominator in the last term is $x-a$ instead of $a-x$. This leads to the modification of the hyperbola equation.

$$\frac{b}{a}\frac{b}{a} = \frac{y}{a+x}\frac{y}{x-a}$$

$$\Rightarrow \quad \frac{b^2}{a^2} = \frac{y^2}{x^2-a^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Thus, all conic sections are determined b°ade by constructions and their equations. Pascal's theorem can thus be used as an implicit definition for conic sections.

Chapter 4 Duality

The impulses from Desargues and Pascal disappeared with them from the mathematical landscape. Specifically, Pascal's essay on conic sections disappeared; only a sketch of this exists today. Even °a Desargues' work was completely forgotten, although it eventually reappeared. The analytical geometry founded by Descartes proved very fruitful in many contexts, and with the impulses from Leibniz and Newton, it developed b°ade in breadth and depth throughout the p°a 1700s. Euler's work in particular established a completely new mathematics.
However, towards the end of the 18th century and at°the beginning of the 19th century, a new germination of the other type of geometry took place. In the Ecole Polytecknic founded by Ges- pard Monges, an environment emerged in which pure geometry was central. Monge himself developed the descriptive geometry that made it possible°to create images of three-dimensional bodies, and from a number of students such as Brianchon, Gergonne and Poncelet emerged what would become projective geometry. Although they were°initially unfamiliar with the work of Desargue and Pascal, the most important principles were found and several new ones were added. This applies to principles such as duality and projectivity.

4.1 The duality principle

35. When considering Desargues' theorem in the introduction, we noted the great symmetry in the theorem, which was evident in the fact that there were ten points and ten lines in the configuration. We do not see the same symmetry in Pascal's theorem. Here, in addition to the conic section, we have nine points, but no more than seven lines. h °a, however, there is a configuration in which the number of points and lines are reversed; here there are nine lines and seven points.

Image 21***: Brianchon's theorem***

Figure 4.1: The duality principle

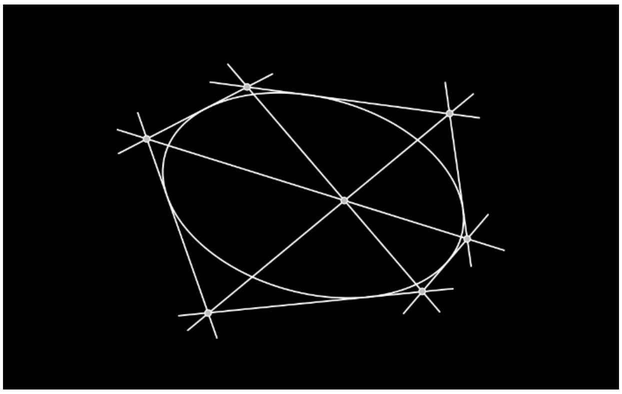

Figure 4.2: Brianchon's theorem

Given a conic section and a hexagon that circumscribes this. The three main diagonals in the hexagon will then meet at the same point.

The theorem is called Brianchon's theorem, and is possibly even simpler in expression than Pascal's theorem.

36. We call Brianchon's theorem as *the dual* of Pascal's theorem. This is not only because the number of lines and points is reversed, but because the entire structure is reversed. In both sentences we take a conic section as our starting point, but where we place points on the periphery of one, we place tangents to the other. Tangents to a conic section are thus the dual image of points on the conic section. Furthermore, we draw lines between points, while in the dual case we find points between lines. This also continues as; in Pascal's case we find three points between three pairs of lines, while for Brianchon

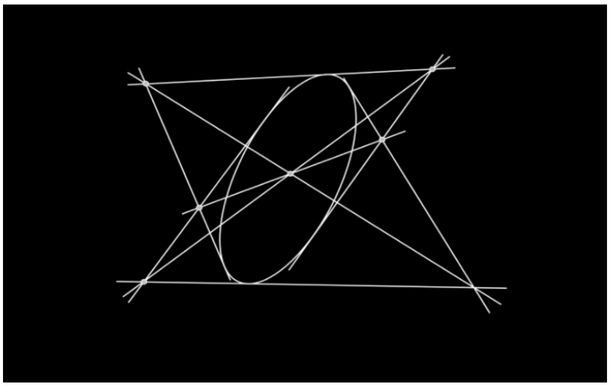

Figure 4.3: Pappos variant Brianchon

draws three lines between three pairs of points. Finally, we draw a line between three points and find a point common to three lines.

37. The basic duality operations are thus that points become lines and vice versa, and that lines through points become intersections between lines. There is another reason for° introducing the elements at infinity; ordinarily we always find a line through two points, but for lines we do not find a common point n°ar they become parallel. With the introduction of points at infinity, duality applies.

38. The conic sections appear°as self-dual; on°a the periphery we can have b°ade points and lines. Another aspect of the self-duality of the conic sections is that°a Bri- anchon's theorem also°a becomes Pappo's theorem. Pappo's theorem is dual to itself; the structures consist°of nine points and nine lines in a symmetrical relation. While Pascal's theorem changed to Pappo's when the conic section became two lines, Brianchon's changed to° when the conic section became two points. This happens when an ellipse becomes increasingly narrower, and finally g° ar together into a line segment. To°a this m°a we choose a variant of Brianchon that allows this; there m°a be no diagonals where the conic section becomes a point, but the diagonals m°g°a to other intersections between the lines.(Fig.4.3. Pappos arising°ar can be expressed dually: Given two points, and three lines between each point. Then three diagonals are formed between the intersections that meet at the same point.

39. The fact that the conic sections in°this way can become two lines on°one side and two points on°the other is a key property. It will°play an absolutely decisive role in°how the various images are metamorphosed. The principle of duality, which is initially°presented as an abstract principle, turns out°to be able to be understood°morphologically in that the conic sections have this dual possibility of transformation.

40. Both point-line structures; Pappo's and Desargues' theorems are thus self-dual. The simplest line-point structure, the triangle, is also self-dual; it can either be regarded as three points with three common lines, or as three lines with three common points.

41. According to one tradition, dual theorems and relationships are set up in parallel columns. Even if we do not follow this, we can summarize the basic relationships in this way.

• point	• Line
• Two points make a line	• Two lines give a point
- A triangle best ar of three points and three common lines	- A triangle best ar of three lines and three common points
- There are points on in the conic section	- There are lines at for the conic section
- A conic section can become two points	- A conic section can become two lines

42. Theorems can also be set up in this way, and in purely practical terms, one can swap a lines and points and create a new theorem. We can look at an extended La Hire in this light; we take this as our starting point and swap it around literally.

Given a conic section, a *point outside* the conic section, the two *tangents* to the conic section, and *the polar* common to *the tangent points*. We draw two new *lines* from *the point above* the conic section. We draw *lines* between *intersection points* with the conic section, and these have a common point with *the polar*.	Given a conic section, *a line across* the conic section, the two *intersections* with the conic section, and *the pole* common to *the tangents*. We add two new *points* on *the line outside* the conic section. Between *the tangents* to the conic section we find *points*, we draw *common lines*, and these have a common *line* with *the pole*.

4.2 Development of Brianchon's theorem

43. In the same way as Pascal's theorem, we can form pentagons, squares and triangles dually. We could do this in a purely abstract way by taking the already formed Pascal specializations as a starting point, and following a

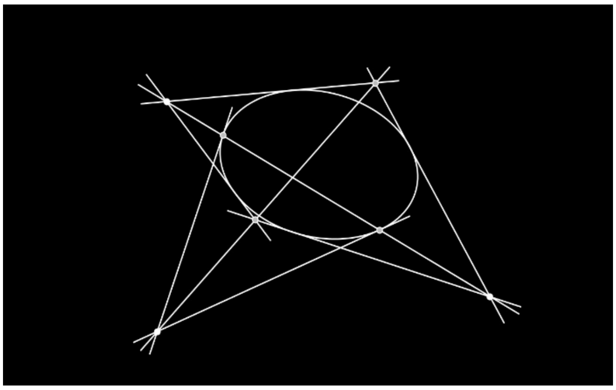

Figure 4.4: Dual extended La Hire

principled dualization like the one we have described above. However, we will stick more closely to the morphology of a see also ap a the dual formation process.

44. The special Pascal configurations arose when two points on a periphery coincided to form a point, and the connecting line between them became a tangent. The dual process is that two lines approach each other, and eventually coincide. The infinitesimal process then tells us that the point becomes the tangent point where the tangent is tangent.

Axiom 2. *When two tangents pa a common conic section merge into one, the common point between them becomes the tangent point between the tangent and the conic section.*

45. All the variants from Pascal are completely dual to this one. If we allow a couple of lines to coincide, we get the following theorem.

Figure 22*: Given a conic section circumscribed by a pentagon. We find two diagonals in the pentagon. A line through the fifth corner of the pentagon that passes through the opposite tangent point will also pass through the intersection the diagonals.*

This statement is useful because if a conic section is given by five lines, we can find the points of tangency between the conic section and the lines, and thereby find five points for the conic section.

46. As in Pascal's case, we have two variants of squares that arise when two pairs of lines coincide.

Image 23*:* ***Dual Mclaren***
Given a conic section, and a square that circumscribes this. A line between two

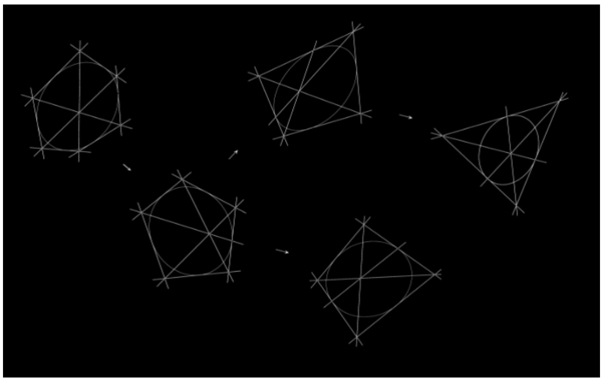

Figure 4.5: Variants of Brianchon's theorem

of the tangent points will then go through the point where the diagonals of the square meet.

47. The second square statement is given by:

Figure 24: Given a conic section, and a square that circumscribes it. Two lines between tangent points and corners in the square will meet on diagonal in the square.

48. Finally, we have the dual of the triangle theorem.

Figure 25: Given a triangle and a conic section inscribed in it. The lines between the corners of the triangle and the tangent points will meet at the same point.

These developments are shown in figure 4.5

49. As for Pascal's theorem, we can derive several relationships also from Brianchon's theorem. We obtain constructions, metric relations and equations. Also a here we can find the different variants by morphological movements, or we can translate the constructions directly by a dualizing the Pascal variants.

4.3 Dual constructions

50. By using Pascal's theorem, we could construct conic sections as point curves. By using Brianchon's theorem, we do the dual; the conic sections appear as line curves, or envelope curves. By finding a number of tangents to a conic section based on Brianchon's structure, the curve is formed.

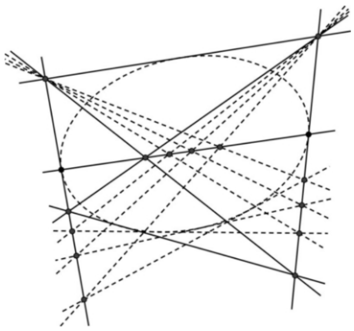

Figure 4.6: Locus of ellipse

51. The clearest construction picture is obtained by starting with the dual McLaren. The theorem states that if a conic section is circumscribed by a quadrilateral, then the lines between opposite points of tangency will pass through the intersection of the diagonals. We then start with three sides of the square and insert a diagonal. From a point on the diagonal, we draw lines to the corners, and through the points where the diagonals meet the opposite side, we draw a line; this is a tangent to the conic section. Continuing the process produces an ellipse.

52. The construction of the parabola can be obtained by a modification of McLaren. We leave the two tangents as a "V", and let the upper tangent g a to infinity. The infinite ellipse g is transformed into a parabola, and we f ar the regularity:

Figure 26: Given a parabola, two tangents to it, and the line between the tangent points. We find parallels to the tangents that meet on the transversal, and where these meet the tangents we draw a line. This is also tangent to the parabola.

If we let ten a point p a line move, the new tangent will envelop the parabola.

53. If we allow the upper tangent to move even further, so that it comes from below, the conic section will be a hyperbola. The construction of the hyperbola proceeds ar p a the same way as for the ellipse.

54. As for the Pascal variant, we can also a here let the tangent points g a to infinity. the two tangents will then be asymptotes. The transversal will also g a to infinity, and points p a this will give parallel lines, resulting in the hyperbolic image:

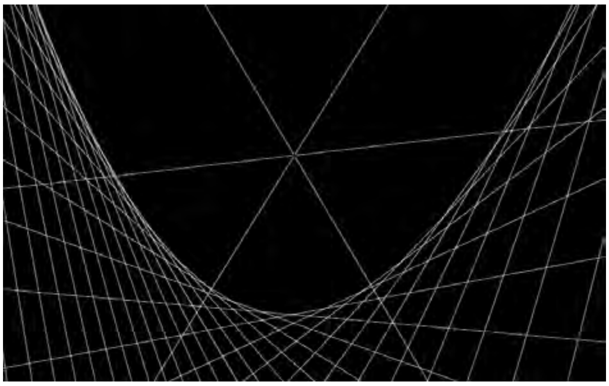

Figure 4.7: Envelope Parabola

Figure 27: Given a hyperbola, its asymptotes and two tangents. The lines through the points where the tangents meet the asymptotes are then parallel.

We construct the envelope hyperbola by having a tangent. From the meeting points with the asymptotes, we draw parallel lines, and where they meet the asymptotes, we draw tangents. By moving the parallels, we find lines that envelop a hyperbola.

4.4 Metric conditions

55. As in the case of Pascal, we also find metric relationships in the point line structures that arise, and this applies in particular when a r parallelism arises. We show a relationship for each conic section.

56. The starting point for the ellipse is McLaren. If we allow the two tangents to the main axis to be parallel, and the transversal itself to be perpendicular to these, this will become the axis of the conic section. If we also let the third tangent be perpendicular to the two lines, the product law for ellipses will also emerge. Three lines through a point intersect two parallel lines so that the conditions are equal. We have:

Metric law 7. Given a conic section, a major axis, and the tangents to the ellipse where it meets the major axis. A tangent intersects the tangents at two points, and the product of their heights på the axis is constant.

$$\frac{a}{c} = \frac{c}{b}$$

$$\Rightarrow from = k$$

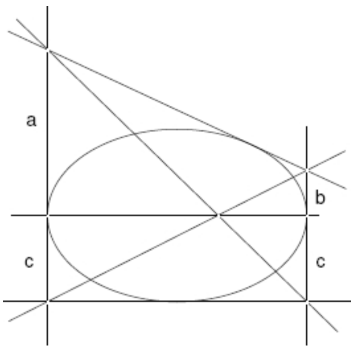

Figure 4.8: Product law ellipse

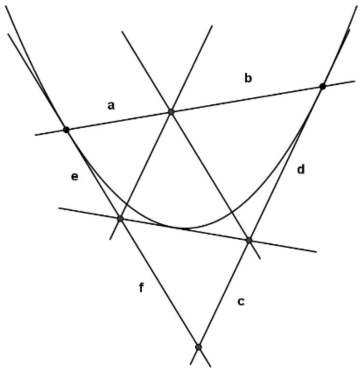

Figure 4.9: Ratio law parabola

57. A known theorem metric theorem for the parabola also applies. Based on the parabola's envelope theorem (26), a known metric relation associated with it emerges.

Metric law 8. *Given a parabola and three tangents to it. Then one tangent will intersect the other two so that the ratio of lengths will be equal.*

$$\frac{c}{d} = \frac{e}{f}$$

58. We see from fig.4.9 that both ratios are equal to the ratio $\frac{a}{b}$ på line between the tangent points.

59. The hyperbola theorem above immediately gives a metric theorem. The two asymptotes, tangents and parallel lines form a trapezoid, and this contains two equilateral triangles. This gives a product theorem:

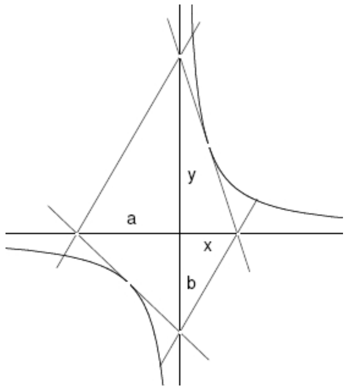

Figure 4.10: Product law hyperbole

Metric law 9. *Given a hyperbola symmetric about the x and y axes, and a tangent to the hyperbola. This intersects the axes at points, and the product of the distances to the origin is constant.*

$$\frac{y}{a} = \frac{b}{x}$$

$$\Rightarrow y = \frac{k}{x}$$

4.5 Envelope equations

60. The metric law we have found for the hyperbola is also an envelope equation for it where the axes in the coordinate system are the asymptotes of the hyperbola. Here the coordinates are the intersection of the lines with the axes. The envelope equations for the curves n˚ar they are symmetrical about the axis, p˚ar similar m˚ ate as we found the curves from Pascal.

61. For a the ellipse's equation, we start from the metric law.

Metric law 10. *The envelope equation for a central ellipse with axes a and b is given by*

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$$

We then see from the figure

$$\frac{y}{x} = \frac{y'}{x-a} = \frac{y''}{x+a}$$

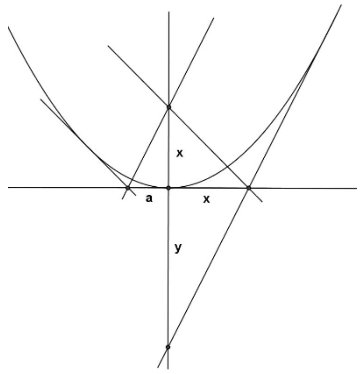

Figure 4.11: Envelope parabola But

we have that $y' \cdot y'' = b^{(2)}$ so that

$$\frac{y}{x}\cdot\frac{y}{x} = \frac{y'}{x-a}\cdot\frac{y''}{x+a} = \frac{b^2}{x^2-a^2}$$

$$\Rightarrow y^2x^2 - y^2a^2 = x^2b^2$$

$$\Rightarrow \frac{a^2}{x^2}+\frac{b^2}{y^2}=1$$

We see that the equation has the same form as the point equation, but the numerator and denominator have switched places.

62. The equation for the hyperbola can be found in the same way. The equation for the hyperbola has the same form as the equation for the point version, $y= ax^2$. We find it by setting the McLaren pa upright as we did for the point version, and here is the geometric picture.

Figure 28: Given a parabola and three tangents to it. From one tangent point we draw a diameter, and where the tangent intersects the other lines we draw parallels to the remaining tangent. These will meet at the diameter.

63. For the forward equation, we let one tangent be the x-axis, the diameter the y-axis, and one tangent forms a 45^0 angle with the axes. From fig.4.11 we realize equilateral triangles, and the equation is found.

$$\frac{y}{x}=\frac{x}{a}\Rightarrow y=\frac{1}{a}x^2$$

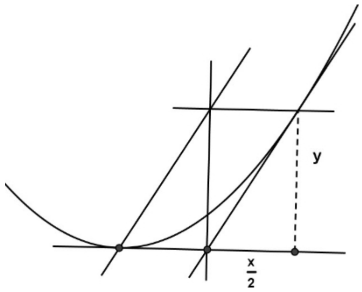

Figure 4.12: The slope of a parabola

64. The gradient of the elementary parabola is easily obtained by the triangle variant of Brianchon. N°f we have a parabola, and one tangent is the line at infinity, we have:

Figure 29: Given a parabola and two tangents to it. Through the tangent point we draw parallels to the other line; these meet at the diameter through the common point.

N°hen one of the tangents is the x-axis, we see the relationship.

Chapter 5

Perspective and projection

With the emergence of projective geometry at the beginning of the 19th century, the principle of projectivity in particular became decisive, and this type of geometry is today known by this name, projective geometry. Although at that time there were other designations, such as synthetic geometry and others, this designation remained. Much of the reason for this was Poncelet's fundamental work, which was called projective properties.

In Poncelet's work, the concept of mapping became central; a concept that would prove to be one of most important concepts in mathematics. Towards the end of the 1800s, the right way was found to algebraize geometry in what became linear algebra. With this, the various geometric representations can be performed uniformly and easily. Such transformations form the basis of all visualization of geometric objects, and changes of these in space.

It is beyond var frame a into pa these transformations, but will later see something pa the metric principles underlying algebra. Var main concern is the geometric images that sadan.

5.1 conic section in perspective

65. The development of the projective principles ga r very broad sa of a r intention here is not g a into p a this in particular detail. What we will see on a are certain basic images, images that also a belong within morphology. One basic image has a to do with what we mean by perspective.

66. Just as Desargues' theorem describes two triangles in perspective, here we will see two conic sections in perspective. We can say that Desargues configuration as a whole can be differentiated into two triangles, perspective point and perspective line. In the same way, we will here arrive at an image that carries the perspective in it, but considered as an image it is part of the morphological landscape.

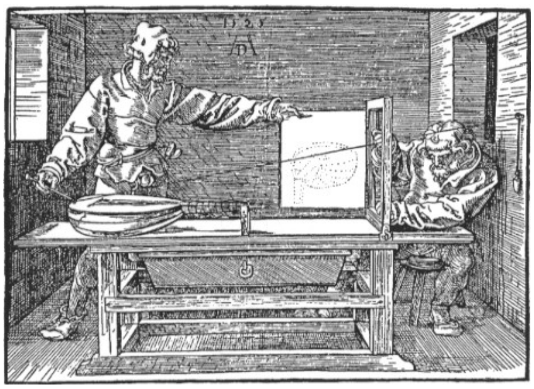

Figure 5.1: Tek

67. The most common form of conic section in perspective is a circle seen from the side. A cup seen from above is circular, but seen from the side we see an ellipse where all parts are pressed together equally. This relationship between the ellipse and the circle makes it possible to easily derive the equation for the ellipse from the equation for the circle.

68. In purely geometric terms, we can also specify this change in shape. We have then given the circle, and a diameter in it. Normally on the diameter we draw lines, and these meet the circle at points. An ellipse with the diameter as its axis divides all the chords in the circle equally. This is shown by the fact that if we connect two points on the circle with a line, and the two corresponding points on the ellipse with a line, then these will meet on the axis.

Figure 30*: Given a circle, a diameter, and an ellipse with the diameter as the axis. We erect two normals on the axis, and where these meet the circle or ellipse, we draw lines that meet on the common axis.*

The ellipse here is a perspective image of the circle.

69. We can use this relationship to construct the ellipse in perspective. We have then given the circle, and a point på the ellipse. Through this point we drop a normal p to the axis. We raise another normal p to the axis, and through the intersection of the two normals with the circle we draw a line that meets the axis. From the point, we draw a new line to the point of the ellipse, and where this meets the second normal, we have a new point på the ellipse. By varying the normal, all the points på the ellipse appear.

70. Another type of perspective is that we see things reduced in distance. Here we have a similarity mapping. We imagine two similar ellipses and their two common tangents meeting at a perspective point. We then have the following relationship:

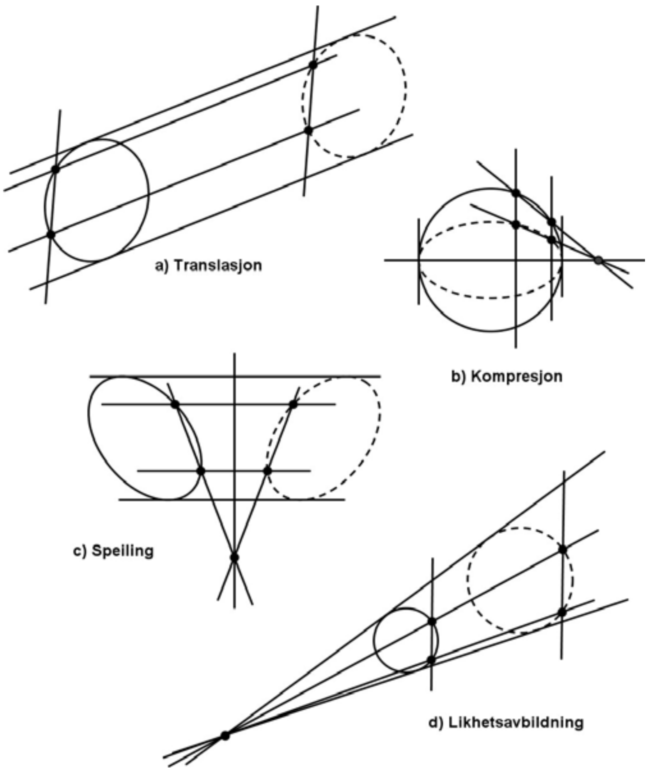

Figure 5.2: Variants of perspective

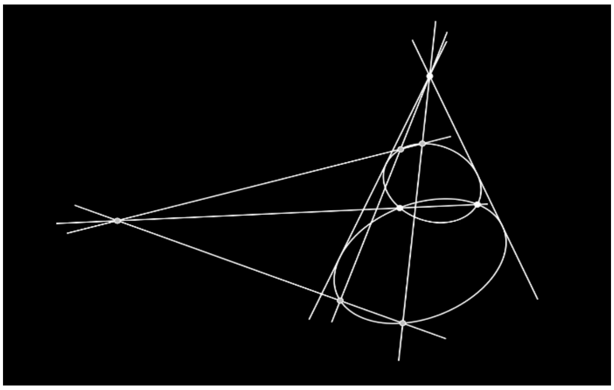

Figure 5.3: Perspective image

Figure 31*: Given two equilateral ellipses in perspective. We draw two lines from the perspective point, and lines through the points pâ each, will be parallel.*

As above, we can construct an enlarged ellipse from a given perspective point, and a given point p°of the new ellipse.

71. We can now°a realize that the last image also°a has a different axis. If we draw lines on°the opposite side, we will see that matching lines meet on°line between the ellipses. W°hen the ellipses intersect, this perspective line will go°through the intersection points. We are not°dealing here with an actual similarity mapping, the ellipse is also reversed.

72. In the last depiction, we are a°dealing with b°a perspective point and a perspective line. This is the general case, and in°this way there is always a continuous mapping from one conic section to another. This is based on the general geometric theorem:

Figure 32*. Given two conic sections, a diagonal between them, two joint tangents to these and a perspective point. Through the perspective point we draw two lines, and where these intersect the conic sections we draw diagonals, and these will meet on the diagonal between the conic sections.*

We call this image *the perspective view*. Lines between two intersections of conic sections are called *diagonals*. Similarly, we call the lines between intersection points formed by a conic section and a double line. Later on, we will use the term *common points* for points formed between common tangents.

73. From this, we can construct a general image of a conic section n°ar we have given the conic section to be imaged, the perspective point, the perspective line,

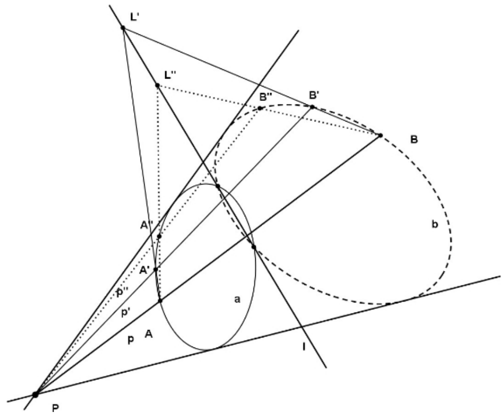

Figure 5.4: Perspective construction

and a point på the new conic section. In fig.5.4 we have given the conic section *a*, the perspective point *P* , the line *l* and the point *B* på the new conic section *b*. Ifå draws the line *p*, we find the point *A* that is in perspective to *B*. We draw a new line *p'* that gives *A'* på *a*, and we will find the perspective point *B'* på *b*. We draw a line *AA'* and this gives the point *L'* på *l*. We draw så *L'B*, and where this meets *p'* we have *B'*. This is how we continue to find the image *b* of *a*.

74. The different perspective variants fåre all based on this basic image. (Fig. 5.2) In translation we have når både perspective point and perspective line are at infinity, while in similarity mapping only the perspective line is at infinity. When the perspective point is at infinity and the perspective line is the polar to this, we have compression. Mirroring also occursa with a perspective point at infinity, but here the perspective line is normally på the lines from the perspective point, and a point på the new conic section is as far from this line as the corresponding på the original.

75. Projective geometry is in its essence a continuation of the starting point of conic section theory, and we realize that the image we have here can be regarded as a spatial phenomenon, where the conic sections are formed by different sections of the plane. Let's take a closer look at this.

76. We imagine a cone and cut across this cone with two planes. This results in two cone sections, but we also have a line formed between the two planes. This line will necessarily pass through two points of intersection between the conic sections. We therefore call this line a *diagonal* to the two cone intersections. The vertex pof the cone can be called a perspective point. In the image we have created, we add a new plane through the perspective point. This plane

will cut the cone in two lines from this point, and it will also °cut the other two planes in two lines. The three lines formed will then meet at the same point. From this realizes the perspective theorem.

77. We have already seen several variations of the image, and we will look at °a few more. By °letting the two conic sections approach each other, we can eventually let them have common tangent points with the line. Here, the two conic sections will also °a tangent to each other at two points, and the diagonal between them will be the line between the two tangent points. For both conic sections, the diagonal and the perspective point then become a pole and polar, and we can call these a *common pole* and *common polar* for two conic sections that are tangent to each other at two points.

Figure 33. *Given two conic sections that are doubly tangent to each other, and a common diagonal, two common tangents and a common pole. Two lines through the common pole intersect the conic sections at two points each, and lines through these points will meet at the common pole of the conic sections.*

This theorem is further specialized into the extended LaHire theorem n °a r the two conic sections fall the same, and we f °a r the ellipse image of a circle n °a r the perspective point is at infinity.

78. A particular shape arises °a r n °a r the two lines across the conic sections g ° a r towards the tangents. As they coincide with these, the diagonals between these and the conic sections become the poles of the perspective point.

Figure 34. *Given two conic sections in perspective. Then the poles of the perspective point with respect to the conic sections will meet on the diagonal between them.*

Here we can also °talk about the lines between the tangent points instead of polars, because the perspective point has lost its significance in the configuration (Fig. 5.5).

79. The two cone intersections between the lines can also °be positioned so that they intersect at four points. It is then possible °to draw as many as six diagonals between them. In a concrete context, however, only two of these are significant, and we will be able to see which by °looking at the diagonals that emerge from different planes. We will later see that there are also °a morphological reasons for this, but in this context it is essential that we become aware °a that we can form two different diagonals by °a drawing lines from the perspective point.

80. By °drawing lines through the perspective point, and finding lines through the intersections with the conic sections, we can also °find two diagonals n ° w h e r e the conic sections only intersect at two points, and n ° w h e r e they do not intersect at all. For the time being, we can define the diagonals in this way °and n ° where they do not .

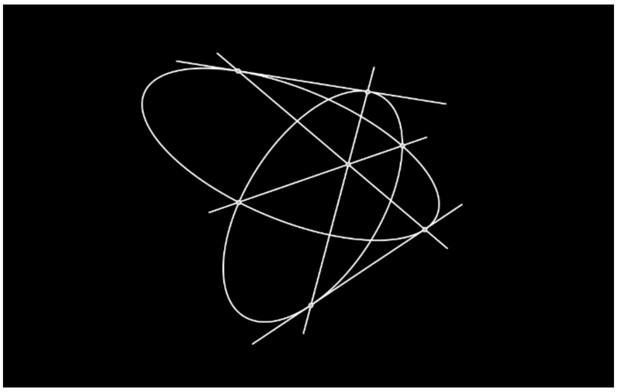

Figure 5.5: Perspective lines as tangents

81. An important perspec- tive transition is to find a circle that is per- spective to a given conic section. In this way, problems related to conic sections can be transferred to circle problems, and once they have been solved here, they can be returned to the conic section. For example, the classical problem of finding the intersection between a line and a conic section given by five points can be solved in this way. (This problem can also be solved in a number of other ways and the applicability of various theorems will be shown here).

82. If a conic section is given by five points, we can choose any circle through two of the points as a perspective circle. We then already have the percentile line between them, and to find the perspective line we use a specialization of the perspective approach above. We add a tangent to the key intersection in one of the points (Construction-.) that is not shared with circle. Where this tangent intersects the perceptive line, we draw a tangent to the circle. The two tangent points will then lie on line through the perspective point. We repeat the process and find the perspective point.

5.2 Dual perspective

83. We have created perspective images of conic sections by a imaging each point on a conic section on a another. In dual perspective, we transfer all lines p a one conic section p a the lines to another. The considerations we have made p a far become visually clear when we imagine them in space. In the dual case, this is not s a clear, and we must a make purely ideal considerations.

84. The fundamental theorem for dual perspective is formed completely dual to the perspective theorem.

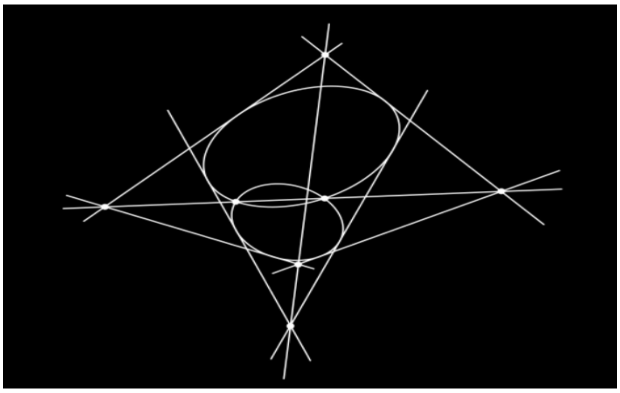

Figure 5.6: Dual perspective

Figure 35. *Given two conic sections and a diagonal to these. From two points on the diagonal, we draw tangents to the conic sections, and through the intersections between these, we can draw lines that go through a perspective point to the conic sections.*

We call this the dual perspective approach.

85. A dual perspective is given by a conic section, a perspective line, the perspective point and a tangent to the new image. While we started the perspective line to find a perspective image, here we start the perspective line. We then follow a completely dual process to the one described, and an envelope image for the original conic section is created.

5.3 Projection

86. Projection is in a sense a repeated perspective. That is to say, an image is repeated; that we find a new perspective from an image. This second image is in a projection of the first, and it differs from pure perspective.

87. We can define projection for entire geometric images, but we see the principle more simply when mapping a line onto another line. In this mapping there is no new shape, a line remains a line. However, we can identify the individual points from the image with the points p of the original image. We say that those points correlate that have a common line through the perspective point.

88. If we repeat in a this depiction by selecting a new perspective point and a new line, then all the points p a will be depicted again. These new points are not

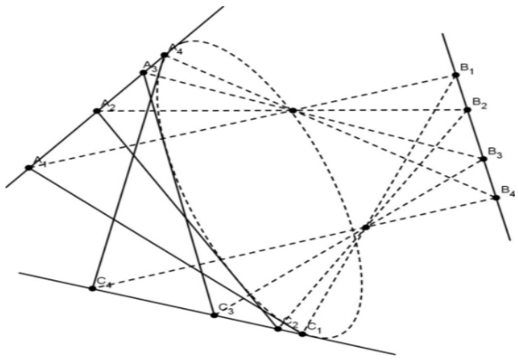

Figure 5.7: Projection

are not perspective to the first anymore, we do not find any common point connecting correlated points pa the two lines. Here we say that the points pof the last line are a projection of the first line.

89. However, it turns out that the connecting lines between correlated points all lie on a conic section. This is a definition of a conic section in projective geometry. In morphological geometry we do not have this starting point for conic sections; here we must show that what emerges must be a conic section based on the geometric images we develop.

90. In what appears above, the Brianchon theorem is present. We first arrange some p a order above. We start with a line a, and project its points to the lines b and c above the points P and Q. The fact that we make a perspective both ways is equivalent to repeated perspective. We choose a point A p a a, and this gives the points B and b, and C p a c. Through the points P and Q g the lines p and q. The connecting line r between B and C should a lie on a conic section. When we move A we realize that certain lines ma belong to the lines in the projection. When A meets the line b, a line from here to B will belong to the conic section.

91. Thus we see that Brianchon's theorem justifies that correlated points lie on a conic section. Conversely, if we define a conic section based on correlated points, then we can justify Brianchon's theorem pin the way we have seen.

92. We can also a do the dual construction; here we have lines through a point that are depicted as lines through another point via a line. We see from Figure 5.8 that the lines b_n are perspective to the lines a_n. We repeat this and then the lines c_n become perspective to the lines b_n. The lines a_n and c_n are na in projection through the points *A* and *C* pa the conic section that mediates the projections. Integer

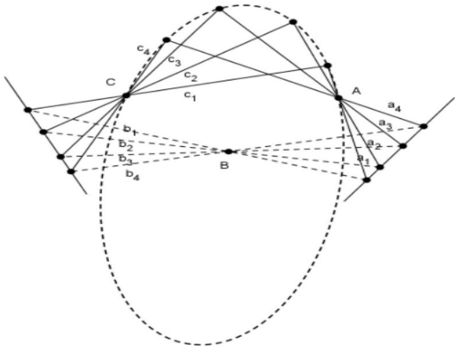

Figure 5.8: Dual projection

Based on the method°above, we can justify that we have °a conic section in Pascal's theorem.

5.4 Projections as group

93. What happens if we go °further, and let the points we have found be depicted on° yet another new line? Then it turns out that we are still°dealing with a projection; a projection of a projection is still a projection. Projections form p°a this way° ate a group, that is, we can always find a projection corresponding to the compositions of two.

94. These conditions are clearly evident in the algebraic methods associated with projective geometry. Here the elements are expressed by s° akal projective coordinates, which are vectors with three elements. A projection is expressed here as the result of the vector multiplied by a matrix. Repeated matrix multiplication still results in a matrix, and thus the group property appears.

95. This group property can also be shown°by the preservation of the so-called° two-belt relationship [1]. It can be shown that this applies to a perspective, and that it

[1] N°ar points are transferred perspectively from one line to another, all distances between the points are changed, but a certain ratio is °ar; the double ratio. If °four points A_1, A_2, A_3 and A_4 p°a a line a, are transferred to points B_1, B_2, B_3 and B_4 p°a line b, then the distance from A_1 to A_2, divided p°a by the distance from A_2 to A_3, multiplied by the distance from A_3 to A_4, and finally divided p°a the distance from A_4 to A_1 will be the same size as that between the points p°a b.

$$\frac{A_1A_2 \cdot A_3A_4}{A_2A_3 \cdot A_4A_1} = \frac{B_1B_2 \cdot B_3B_4}{B_2B_3 \cdot B_4B_1} \tag{5.1}$$

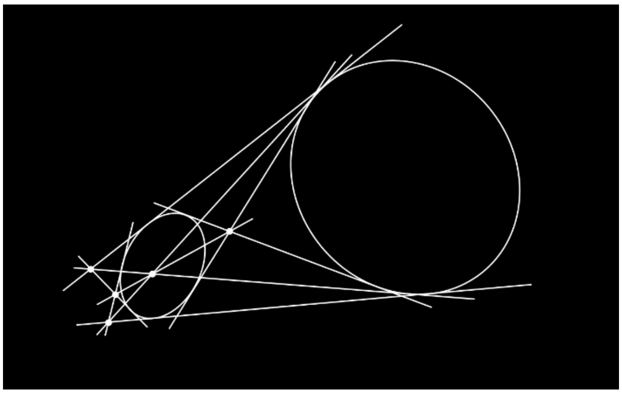

Figure 5.9: The projection theorem

therefore also °a applies in a projection. Therefore it applies to a repeated projection. This means that we also °a have the following metric theorem:

Metric law 11. *Given a conic section, and two tangents a and b to it. Another four tangents transfer points from line a to line b. Then the double ratio between the points på of the two lines will be the same.*

By °moving the keys in °different ways °we can get °many special variations of this. We °won't go into °this here, but some tasks are given.

96. We can also °a justify the group property of projections purely geometrically. We then use the following extension of Brianchon's theorem.

Figure 36. *Given two conic sections that do not intersect, with two common outer tangents. We add a point p °to each of the tangents, and from these points we draw tangents to the conic sections. These intersect at two points whose common line g ° passes through the inner common point between the conic sections (Fig. 5.9).*

We call this the dual projection theorem. The image g°ar over to Brianchon's theorem n °ar one of the conic sections g °ar together into point pairs.

97. We study figure.5.10 to °see how the above theorem continues to project. Here we have three lines a, b and c. From the point P we draw the line p, and it leads the point A p °a the line a, to the point B p °a the line b. We adds °a a tangent q p °a the ellipse Q, and this leads B to the point C p°a the line c. This compound operation results in the point A g° ar to the point C. However, this can be done by the tangent r p°to the ellipse R. But the formation of the lines r p°in this way is an envelope construction of R based on the dual projection theorem. This shows that a combination of a perspective image and a projection becomes a new projection.

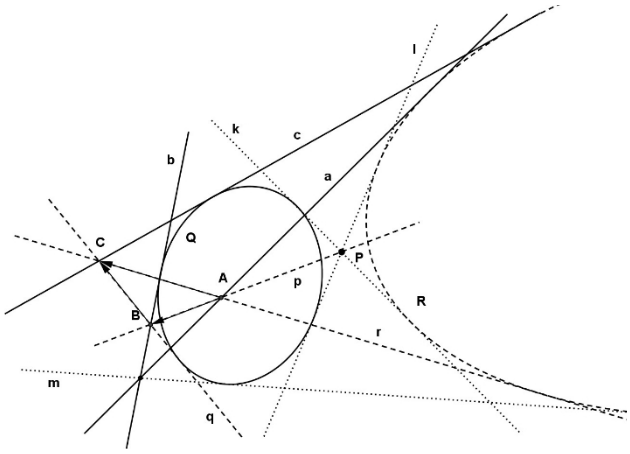

Figure 5.10: Projection

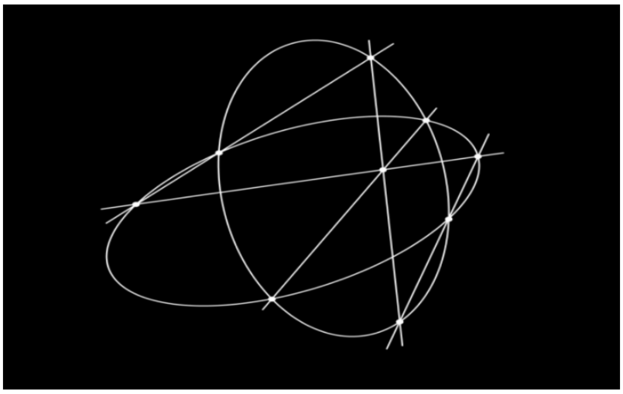

Figure 5.11: The projection theorem

98. We can make the same considerations in the dual case. In this case, we use an extension of Pascal's theorem:

Figure 37. Given two conic sections that intersect at four points. We draw lines through two of the intersections, and these intersect each of the two conic sections at a further point. We draw lines through the points of intersection ɑēach of the conic sections, and the two lines then meet ɑt̄he diagonal between the other points of intersection of the conic sections.

We call this image the projection theorem. we justify n ̃a this projection completely dually to the presentation above.

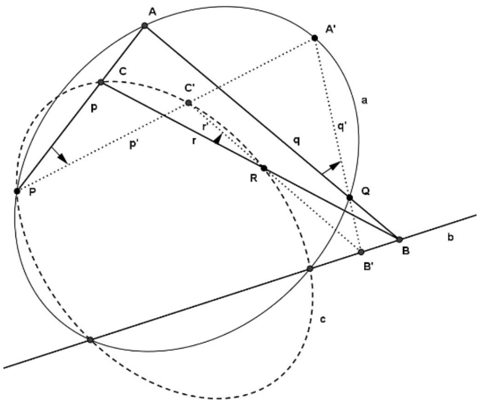

Figure 5.12: Line projection

99. The reason we call the theorem above the projection sentence, and the first the dual projection theorem is that we are used to calling the line images dual. We have also used this principle for naming the perspective sentences.

100. We can see in Fig.5.12 how the dual proceeds. Line *p* moves about *P* , and these are projected to lines *q* about *Q* via the conic section *a*. Furthermore, the lines *q* are projected to the lines *r* through *R* via the line *b*. We see n a that the lines *p* can be projected directly to the lines *r* via the conic section *c*. Thus we see again that a combination of a projection and a perspective is a new projection. (The line QR that hits the intersection of the two conic sections is not drawn in because it does not play a role in the construction.)[2]

101. In total, we have looked at four different sentences in this context. These are the perspective theorem and the dual perspective sentence, and the projection sentence and the dual projection theorem. As previously stated, these are purely phenomenological, as given. But with these as a starting point, we have been able to determine the other images and various relationships. We shall continue along this path and see how new general images arise that explain new relationships.

[2]That a direct combination of two projections gives a new projection can also be shown geometrically. This presupposes a theorem that lies outside the ones we are dealing with. The theorem is: Given three conic sections A, B and C with three common points. Two and two of these have an additional common point that we call AB, AC and BC. We draw a line through AB, and where the line hits A we draw the line to AC, and where it hits C we draw the line to BC. The two lines then meet at a C. Thistheorem is also a specialization of the third degree theorem.

Chapter 6 Imaginary elements

A decisive step in arithmetic and algebra was taken when°the so-called imaginary numbers were taken°seriously. When solving the second-degree equation, we had encountered°a solution with a negative number under the rat sign. It was felt that these numbers had no meaning; there are no numbers that multiplied by themselves become negative. These solutions were therefore called illegitimate or imaginary, and were°generally discarded. S°a the time was ripe enough to introduce a symbol i, which when multiplied by itself becomes -1. It turned out s°a that expressions containing this element could also°a be processed, and calculation rules for this were developed. Even though the imaginary numbers could not be imagined, abstract rules were developed that enabled them to be processed nonetheless.

Poncelet wanted to apply this way of thinking°ate also°a in geometry. In°the same way°that algebra treated unreal elements, and eventually regarded them as self-evident in an extended number system, he would also°treat an extended geometry that also°included imaginary elements. Just as a second-degree equation always has two solutions when°one also°a includes com- plex solutions, so also°a two circles also°a always have two common points, he believed. Two circles that intersect have two real points in common, at the tan- gering the points coincide, and when° the circles do not intersect they have two imaginary points in common.

As an extension of this, Poncelet asserted the so-called°*principle of continuity*, a principle that Carnot had already applied. The essence of this principle is that everything that exists in a geometric image continues°to be there even if the image changes. For example, if we have two conic sections that intersect each other at four points, they will always do so regardless of their position. The four points of intersection are all real n°ar the conic sections g°ar completely above each other. N°ar they move away from each other, they can f°have two real intersections, and then two of the intersections will be imaginary. N°ar the conic sections lie completely

Figure 6.1: Poncelet outside

each other, all four points become imaginary.
It turned out that it was not as easy to draw in imaginary geometric elements as it had been to introduce imaginary numbers. Several people claimed that such elements do not exist; if two crescents do not intersect, then one cannot conceive of any kind of invisible points somewhere that one cannot see. This view was originally put forward by Cauchy, and was repeated by others. However, several others had a different view, such as Von Staudt. We will come back to these considerations, but first we will look at how the assumption of imaginary elements works.

6.1 Imaginary points

102. The introduction of the circle points is justified in a similar way as the inclusion of elements in infinity. Although these do not appear in the configurations, they have an effect in what we have before us. Many connections in geometry are explained by the fact that what confronts us in reality is connected in an imaginary way. To immediately a f a this clearly, we will see p a an example related to circles. Furthermore, we will look at how circles can be understood as specific conic sections based on imaginary elements.

103. We begin with a central circular statement.

Figure 38. *Given three circles that intersect each other mutually. Through the points of intersection of two and two of them we draw diagonals, and these three will meet at the same point. (Fig.6.2)*

This theorem has several implications that we will see in due course. We call it the tree circle theorem.

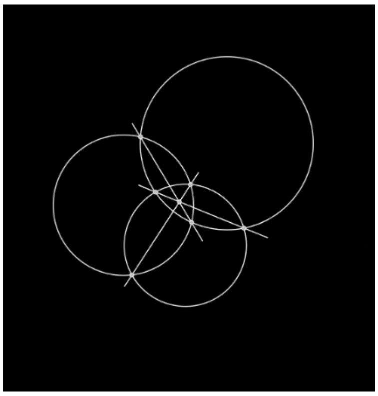

Figure 6.2: Circle sentences

104. We can vary the three-circle set as an image by changing the position and size of the circles. Two of the circles may then no longer . It turns out that the common diagonal of these circles can still exist as a real element. We approach this by the following consideration: Given two circles that than. We let a third circle intersect the two, and find the diagonals between this and the two. The diagonals will meet on a the diagonal between the first two from the theorem above. If we place a new circle over the two original circles, we can find two new diagonals that meet on the diagonal between them. We thus have two points on this diagonal, and can draw it without using the intersection points. When we do this, we see that the diagonal g passes through these points.

105. We can perform the same construction by starting with two circles that do not intersect. By having new circles intersecting this one, we can find points on their common diagonal just as above, and we can thereby draw the diagonal. We can confirm this by finding many points on the diagonal.

106. What we have here is an example of the principle of continuity; what applies in the real case also applies when the image is changed. In the latter case, however, the intersections between the circles have disappeared, but we say n a that they have become *imaginary*. In the abstract, we always say the same thing: two circles have two common points and a common diagonal.

6.2 Circles

107. By considering imaginary points, we will see that the circle can be justified in a similar way as parallel lines. Yet a we have not established the circle as a geometric element; the basic definition

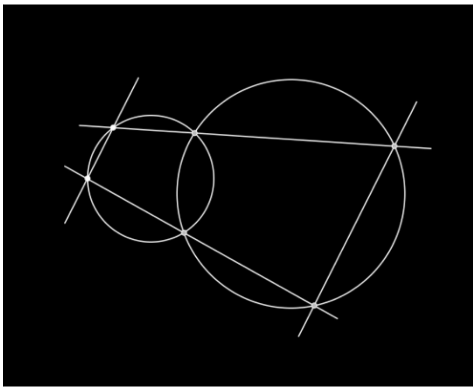

Figure 6.3: Circle theorem

of the circle is basically metric; it is the geometric location of all points that are equidistant from a given circle. How is it determined geometrically? The other conic sections can also a be determined metrically, but we have seen that they also a appear purely geometrically, and that they are determined as distinct by their relation to the line at infinity. What about the circle, can it also arise from purely geometrical considerations? This is the question we will try to answer.

108. The first thing we can realize is that the circle differs from the other conic sections in that two circles can have a maximum of two intersection points, while the other conic sections can have four intersection points with each other. This applies to b ade ellipses, parabolas and hyperbolas, and also a the conic sections' intersection with circles. But the conic sections can also have fewer points of intersection; if they only intersect with each other, they have two points of intersection, and if they are completely outside each other, we see no common points. We will examine the various possibilities in more detail later, but intuitively see that two conic sections generally have four points of intersection with each other, some of which may have collapsed or become imaginary.

109. If we think na the circle as a special conic section and find that two circles never have more than two intersection points in common, then it is natural to ask: where have the other intersection points gone? Since we cannot find more than two real intersection points, two of them must be imaginary, but how are we to understand them? Here we are helped by a particular theorem that shows us the relation of the circle to parallel lines.

Figure 39. Given two intersecting circles and a line through each of the intersections. The circles are each intersected at a pair of points by these lines, and lines through the points become parallel (Fig. 6.3).

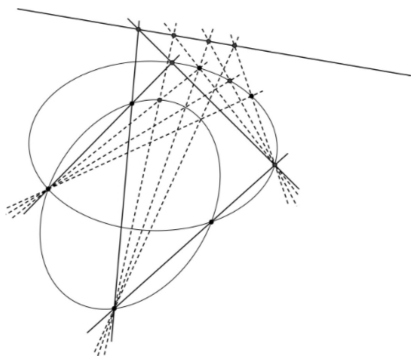

Figure 6.4: Joint diagonal

We call this theorem the parallel theorem.

110. This relationship does not exist for conic sections in general. But the projection theorem (Image**??**) from the previous chapter describes something similar. Here we have two conic sections and draw lines through two of the intersections. These meet the conic sections at two points each, and the lines through these are not parallel but meet on °the diagonal through the other points.

111. We let n °a the two conic sections intersect at just two points. As above, we draw lines through the points of intersection and find the points of intersection with the ellipses, and the lines through these meet at a point. We cannot draw a diagonal in this case, but if we repeat the process, we will see that all the points formed p ° a this m °a t e n lie p °a a line outside the ellipses. This is n °a a diagonal through the imaginary intersections of the ellipses.

112. If we try p °a this m °a t e n °a to find the diagonal between two similar ellipses i.e. two, ellipses of the same shape and direction, we will not find it because the lines through the intersection points are parallel. The other two points of intersection between these will then lie on °a line at infinity, and the line at infinity is then a diagonal between similar ellipses.

113. If we look again at °a the situation with the circles, where the lines also °a become parallel, we realize that the diagonal between the two circles also °a lies at infinity. This is a natural consequence of what we have seen, because all circles are uniform with each other. This °means that two circles always intersect at two points at infinity, and that the line at infinity is a diagonal between them.

114. I f °all circles have the line at infinity as their common diagonal, then a little consideration tells us that the circles m °have the same two imaginary points in common.

We realize this by considering that if three circles are to have a common diagonal, then they cannot have more than two points in common. These imaginary points in the infinite that all the circles have in common are called *the circle points*.

115. We have thus determined circles to be conic sections that g ar through two specific points in infinity. From this we can see how conic section theorems become circle theorems by adding points as circle points, and vice versa, that circle theorems can be generalized to conic section theorems where instead of circles we have conic sections through the same points. In this way, a number of scattered theorems can be seen in a common light

116. This methodology has been the subject of persistent controversy; some have claimed its validity outright, while others have been much more critical. We will look at these issues in some detail, but first look at some immediate consequences of the relationship.

6.3 Tree conic section theorem

117. In the three-circle theorem, we are dealing with three circles through the circle points, and three diagonals between two and two of these meet at the same point. If we let n a the circle points instead be two arbitrary points, these will be common to three arbitrary conic sections, and three diagonals will meet the same point.

Figure 40. *Given three conic sections through two common points. The diagonals formed between the other intersections of the conic sections will then meet at the same point.*

We'll call it the three-cone intersection theorem, and we'll see that it has many important applications. [1]

118. From the relatively special circle theorem, we have thus found a theorem of a far more general character, and it contains several theorems as possibilities. When one of the conic sections stretches out and becomes a pair of lines, we have the above projection theorem, and when two conic sections become two pairs of lines, we have. Pascal's theorem.

119. Instead of letting the two points common to the three cone intersections be circle points, we let the points between only two of them be circle points. Then f we have two

[1]This theorem also appears a as an example from algebraic geometry. Given two third-degree curves $f(x, y) = 0$ and $g(x, y) = 0$, intersect at nine points. A linear combination $h(x, y) = a f(x, y) + b\, g(x, y)$ is also a third-degree curve, and it m a also a be 0 at the common points between f and g. When each of the third degree curves degenerates to a conic section and a line, we have the triple conic section theorem.

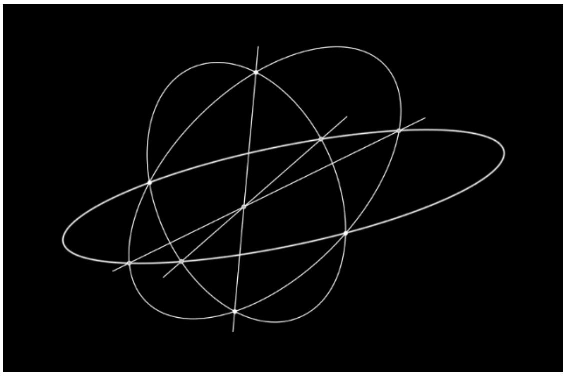

Figure 6.5: Three conic section

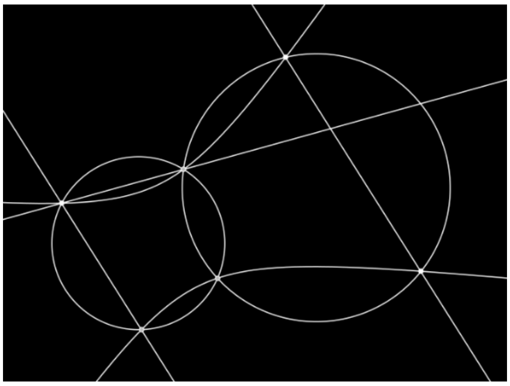

Figure 6.6: General parallel theorem

circles, and the third conic section g ˚ a r through the intersection points of the circles. The diagonal between the two circles is n̊a the line at infinity, and the diagonals between the conic section and the circles m̊a meet p̊a the diagonal between the circle, thus becoming parallel.

Figure 41. *Given two intersecting circles and a conic section through the intersections. The diagonals between the conic section and the two circles are then parallel (Fig).*

This is an important theorem for constructions of different kinds, some of which are given as tasks.

120. The statement above provides a surprising connection to the radius of curvature of a conic section. The connection is found by a modification of the above theorem. Here,˚the two points of intersection between the circles merge into one so that it becomes

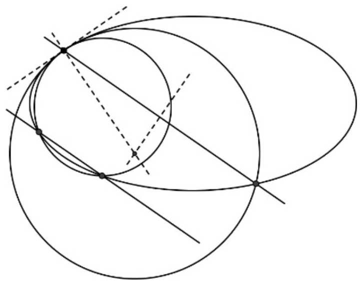

Figure 6.7: Curvature

a point and the tangent between them. The conic section that goes °through the coinciding points will also °touch the circles here and we have:

Figure 42. *Given a conic section, and two circles that are tangent to this at °a the same place, and that also °a intersect the conic section. The diagonals between the circle and the conic section are then parallel.*

121. The radius of curvature of a conic section at a given point is defined as the radius of the circle that has a triple point in common with the conic section here. If we have n°a given circles that are tangent at a given point and vary this, then the diagonal between this circle and the conic section will have the same direction all the time. If we change n°a the circle so that the diagonal also °a g °passes through the tangent point, then we have triple tangent and the correct circle. We can find that direction by an arbitrary °circle, and from there we can find the circle of curvature (Fig. 6.7).

6.4 General considerations

122. We already see that by °generalizing circle points, and moving back and forth between imaginary and real points, theorems can be brought together in °the most surprising ways. A question °that arises °is: is this method rigorous enough; can we talk about imaginary elements that we don't °see in the same way as real ones, and can we generalize circle points as we have done?

123. The fact that the imaginary elements were not originally accepted on °an equal footing with imaginary numbers is due in the first place to °a the fact that the numbers are more abstract in character than the elements of geometry. We are used to °a treating numbers abstractly in algebra, and here we are not dependent °a on imagining a specific size

when processing the various expressions. This is different in geometry, as geometry is basically drawing and viewing. You thus associate the geometric elements with what you are looking at. When some elements become imaginary, they no longer exist in the view, and it is difficult to relate to such elements in the same way as to imaginary numbers.

124. This problem is best ar in one form or another as long as you think of the geometry as what you are looking at. If one thinks of geometry as more comprehensive, things are different. Geometry then appears to us in various ways, of which geometric images are the first. Here we draw and construct, and form an overall picture of the situation. By using equations, which we will not go into further here, we can also treat the imaginary elements directly. What we gain here is lost in that we no longer have an image in front of us. Geometry thus *expresses* itself in a double way, in a visual image on the one hand, in an algebraic way on the other.

125. The third way to do this is to treat geometry abstractly in purely conceptual terms. At this level it is then the case that, for example, two circles always have two common points. We can denote this symbolically by $S_1 \cap S_2 = \{P_1, P_2\}$. These points can be real, in which case we can view them in a geometric image. If they are tangent, they coincide, and if the circles do not intersect, the points become imaginary. Then the points will not appear in the image. The image, or view, thus only covers part of the geometry, the imaginary points are not dealt with here. The same applies to the elements in the infinite, which are also not included in the view.

126. The elements of infinity have often been called "ideal points", i.e. a points one must. In the same way one can consider the imaginary points, they are imagined and do not appear. This consideration is based on the fact that we see the considered geometry as the real one, and in addition to these real elements we have the ideal ones. However, if we consider the nature of the geometric elements, we must say that all geometric elements are ideal. Some of these ideal elements we can imagine, others we can only think, but as we shall see, eventually we have a kind of experience of. Only when it becomes clear that geometry is something more than meets the eye can we freely relate to it as a whole. We then do not seek any hidden elements, but say that the view gives us something, the rest must think and relate to indirectly. Other aspects emerge through algebraic treatment, and still others through abstract treatment. Thus, the ideas of geometry emerge through a multifaceted approach.

127. In visual geometry we therefore operate on two levels. On the one hand, we have the abstract whole or idea that is continuously active, on the other hand, the perceived image that provides the image, but which always appears

In particular. We cannot say that the real geometry is the abstract geometry, or that the real geometry is the perceived geometry; both sides belong together. By the abstract ideas we grasp something that applies to all forms, by the manifold views we experience the essence of geometry in its most manifold forms, we experience its breadth.

128. The second question is whether you can generalize the circle points. We cannot do this immediately, but in our treatment we do it differently, we assume phenomenologically what emerges from the method, and from what emerges we see what can be justified. When it comes to the conic section theorem above, we leave it as the most general for the time being, and conclude from here to the tree circle theorem and the other theorems we have found. We can continue along this path; from circle theorems we can hypothetically assume the general theorems, and infer other conditions.

6.5 Impact of the circle points

129. Two circles in perspective can also be understood as in this perspective. In the general considerations about conic sections in perspective, we saw how matching lines met on one of the diagonals between the conic sections. When we have two circles in perspective then matching lines will either mates on the diagonal between them, or they will be parallel. We realize the latter immediately because one of the circles can be understood as a displacement of the other. From this we also see that the circles have a common diagonal at infinity.

130. There is a generalization of two conic sections in perspective; we can have three conic sections between two lines. Then three diagonals between two and two of them will meet at the same point.

Figure 43. *Given three conic sections between two lines. Then there are three diagonals between two and two of them meet at the same point.*

131. This theorem can be moved in many directions. If one of the lines is at infinity, the conic sections will be parabolas, and then we have:

Figure 44. *Given three parabolas that are tangent to the same line. Then there are three diagonals between two and two of them meet at the same point.*

This can be transformed further, and we can also find more conditions for hyperboles.

132. We now see what happens if two of the three conic sections between the two lines are circles. Then one diagonal can be the diagonal between the two circles, and the diagonals between the conic section and the circle will meet on it. If is the diagonal between the circles, the line will be at infinity far vi:

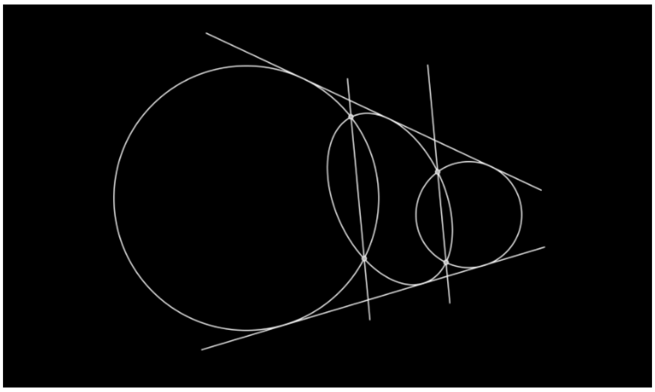

Figure 6.8: Circles and ellipses

Figure 45. *Given two circles and a conic section lying between two lines, so that the conic section cuts across the circles. The diagonals between the conic section and the circles will then be parallel.*

6.6 Monges theorem

133. We have gatt one way; from general conic section theorems we have found several special circle theorems. We will na gar the other way, and generalize a circle theorem to a see what is formed by this

134. Monge's theorem is a central circular theorem, and in way dual to the three-circle theorem. It is not about intersections and diagonals between two circles, but about *common tangents* and *common points*. The outer common tangents of two circles meet at the *outer common point* two circles, and the inner tangents meet at the *inner common point*. When we have three circles, we form three common points, and we have the relationship:

Figure 46. *Given three circles and their three outer common points. These are located on the same line.*

The theorem is a well-known property sof akal homology, or similarity consideration. It can also be applied to the appolonius construction.(Opp..)

135. However, the three inner common points of three circles do not lie on a line, but the line between two inner points always passes through one of the outer common points.

136. A modification of Monge's theorem is when one of the circles approaches the others. The inner common points lie between this one and the others, and when they are tangent we have:

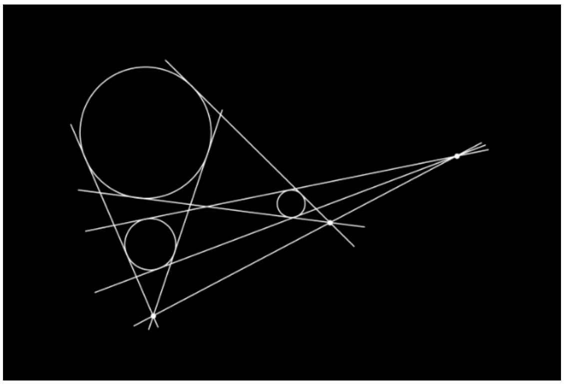

Figure 6.9: Monge's theorem

Figure 47. *Given a circle that is tangent to two others. Then the line through the tangent points will go through common point the two circles.*

From this theorem we can find circles that are tangent to two others.

137. If we regard the circles as special conic sections through the circle points, this leads to a theorem where we have three general conic sections through two points.

Figure 48. *Given three conic sections through two common points. Then the outer tan repeats between two and two will meet in three common points located on the same line.*

138. It may be that we do not immediately see what further processing possibilities lie in this theorem. However, if we look at it more closely, we see that it is the dual of the theorem with three conic sections between two lines. Here we have three conic sections through two points, and we have three common points on a line, as opposed to three diagonals through the same point. Since we have developed different relationships around the three-line theorem, we can by dualize find relationships here. We are not going to do this, but just point out a few key points.

139. A key transformation was that one of the conic sections in L3 became a pair of lines through the perspective point. What is the dual process for this? We imagine one of the conic sections through the pair of points as an ellipse that becomes increasingly narrow, but such that it always extends past the two points. As the ellipse collapses, it will appear as two points on a line through the starting points.

140. If we carry out this process in the image above, we will have two conic sections through two points, and two points on the diagonal between them, and this gives the theorem:

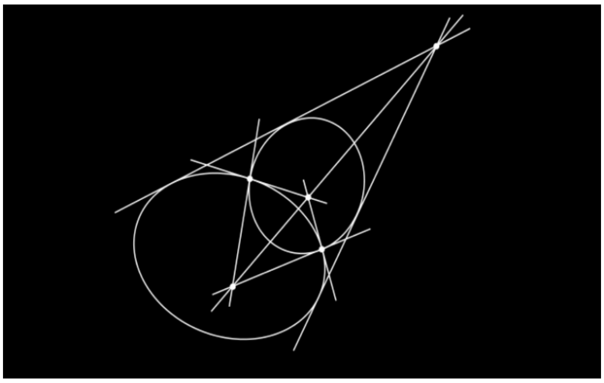

Figure 6.10: Specialization of general Monge

Figure 49. *Given two conic sections, a diagonal through two of the points, and two points on the diagonal. We draw tangents from the points to the conic sections, these form points, and a line here will meet a common point of the two conic sections.*

From this, we can find common tangents to two conic sections if we have intersection points.

141. A further specialization appears when the two points on the lines coincide with the intersections of the conic sections:

Figure 50. *Given two conic sections that intersect at two points. The tangents to each of the conic sections at these points form two points that are aligned with a common point for the conic sections. (Fig.6.10)*

6.7 Ordinary and projective circle

142. Although we have found that circles as we know them can be understood as conic sections that have common points at infinity, we have not yet constructed the circles on this basis. We saw earlier how the parabola arose by calculating the line at infinity and constructing parallel lines. We could already construct the parabola when we knew parallel lines, but is this enough when it comes to the circle? It isn't. Just as we cannot know what actual parallel lines are without a metric, we cannot know when it comes to the circle. Just as in the consideration of Desargues' theorem we saw that we must have given two pairs of parallel lines in order to be able to say anything further about them, so must we have two circles in order to be able to say anything further definite.

143. How do we get sa from the generalized circle time the ordinal? This turns out a not to be possible without metric conditions, just as we cannot determine what two truly parallel lines are without metrics. In pure geometry we have to define what we mean by a pair of parallel lines, and furthermore what parallelism in another direction is, when we do not have the line at infinity. Nor do we have an ordinary circle because the imaginary points at infinity are not something we can really work with geometrically. When it comes to the geometric Euclidean image, analysis cannot help us either; we do not know what orthogonal lines are without having learned it from the world. The peculiar thing turns out then, that the general geometric images we can realize, the Euclidean ma we learn from the world around us.

6.8 Various basic images

144. If we reflect on the basic sentences we have found, will see that we have three different conic section sentences. We have the generalized three-circle theorem where we find diagonals of three conic sections through two points. Then we have seen the generalized Munge's theorem, where three conic sections through two points had three common points on the same line. Furthermore, we have the dual to this; namely, three conic sections that are tangent to two lines and have three common diagonals through the same point. However, we do not have the dual of the extended three-circle theorem, and this is given by.

Figure 51. *Given two lines and three conic sections between them that do not intersect each other. Then the joint keys between them will give three joint points on the same line.*

We immediately see that this is a generalization of two conic sections in connection. This appears when is one of the conic sections above g is combined into a line.

Chapter 7

Focal point and guidance line

The focal points form the s and say the other other kind of entrance to the conic sections. Until now we have considered the conic sections as sections of a cone and as sections of a circle, but just as often, and by elementary definition, the conic section is considered from its focal points. The fact that the planetary orbits are ellipses with the sun one focal point points to the centrality of the focal point. For a long time, planetary orbits were considered to be circular, or compositions of circular motions. This went back to the Greeks, who believed that the circle was the perfect shape, therefore the celestial movements must be circular. We then have the remarkable fact that the focal point is closely linked to the circles, and yet in a kind of polar way.

7.1 Imaginary keys

145. While circles are rooted in imaginary points, focal points are rooted in imaginary tangents. Imaginary tangents are completely dual to imaginary points, and just as imaginary points form real diagonals between conic sections, so do imaginary tangents form real common points between them.

146. To find diagonals between two conic sections with only two trap points, we used the projection theorem (37). The dual construction is a find the common point between two conic sections that g a r into each other so that we only have two common tangents. We can then use the dual projection theorem (36) to find the common point between them. We add points P and Q to the outer tangents, find tangents from these to the conic sections, and lines between intersections will then g a r through the inner common point.

147. The two conic sections above have n a two common imaginary tangents, which meet at a real point. We can apply this in a number of ways, but the essential

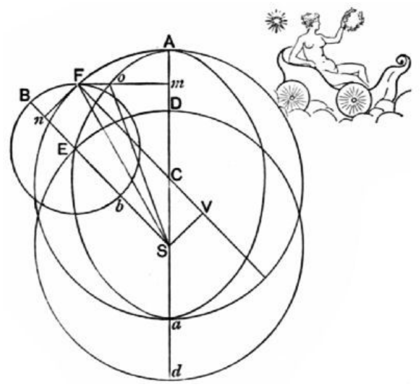

Figure 7.1: Kepler's victory over Mars

To express this, we have n° a r the imaginary tangents g° a r through the imaginary circle points at infinity. Then the remarkable thing happens: the common points become the common focal point for the conic sections. The focal points that play such a central role f °a r thus an elementary interpretation:

Definition 5. The tangents of a conic section from the circle points meet at four points, which we call the focal points of the conic sections.

148. N °i s the definition st °i s as it st °i s, foremost °is the whole thing as abstract, but we will soon see the effects of this. We will again consider the dual projection theorem, but think of the picture in general terms. We let ten °a P and Q p °a lines be the circle points. The outer imaginary joint tangents then form a common focal point for the conic sections. In addition, the tangents to each of the other conic sections form a focal point for each of these. Finally, there are two tangents between the conic sections, which may well be real in the general case. We then have the image:

Figure 52. Given two cone sections with a common focal point, the other two focal points, and two common tangents to the cone sections. Then the common point between the two will beiline with the focal points.

149. At this foremost °a r the dual projection theorems in a completely new form. A variant of this is the simple image.

Figure 53. Given two cone sections with a common focal point that are tangent to each other. Then the tangent point will beiline with the other focal points.

150. Parabolas in the same direction have a common focal point at infinity, which we will explain in more detail as we go along, and which gives a variant of the theorem above:

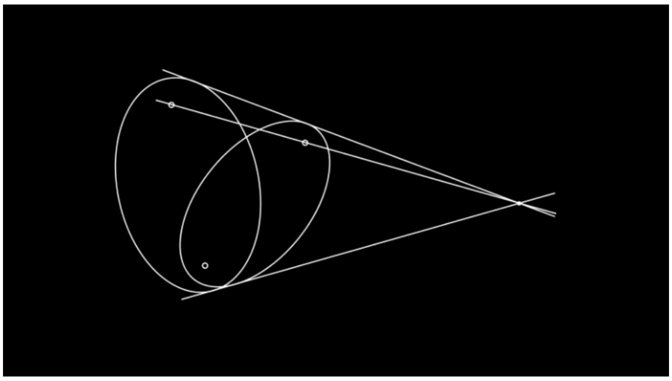

Figure 7.2: Focal point

Figure 54. *Given two parabolas with the same direction that are tangent to each other. In this case, the point of tangency will be in line with the focal points of the parabola.*

7.2 Like angles

151. Using the theorem of two tangent conic sections (53), we can show a key angular relationship to conic sections.

Metric law 12. *The lines from the two focal points to a point on the periphery of a conic section form equal angles with the tangent of the point.*

If we have given a conic section c as shown in a figure (7.3), we can always find a conic section b with both focal points equidistant from the tangent. From the symmetry, we see that the angles are equal.

152. We see here that from the definition of the focal point we can find metric properties in the same way that we found equations and other properties for the conic sections from Pascal's and Brianchon's theorems. We will continue with this, and find key relationships related to lengths.

153. We now start from the perspective theorem (32). Here we have two conic sections in perspective, two lines from the perspective point, and two diagonals between the lines and the conic sections that meet at the perspective line. We let the common bars be the imaginary tangents through the circle points; the perspective point becomes the common focal points for the two conic sections, and we create the image:

Figure 55. *Given two cone sections with a common focal point, and two lines through the focal point. Diagonals between the lines and the cone sections will then meet at the diagonal between the cone sections.*

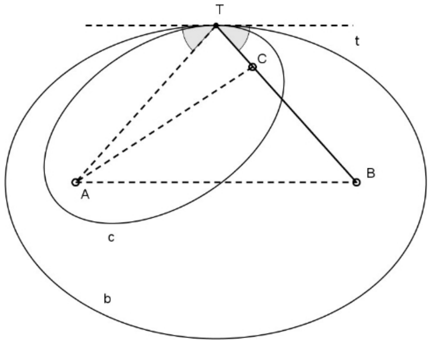

Figure 7.3: Equal angles

154. An immediate special variant is where the two lines merge into one:

Figure 56. Given two cone sections with a common focal point, and a line through the focal point. Tangents where the line meets the cone sections will meet on the diagonal between the cone sections.

155. Before we go any further with, we need to make one thing : *for the circle, all focal points coincide to one, and this is the center of the circle*. We realize this by considering that the circle goes through the focal points, and then the tangents from these points to the circle will be the two tangents in the points. These meet in a point, and this is the pole of the line at infinity, which we have previously determined to be the center of the circle.

7.3 Metric relationships

156. The theorem we will use to find metric properties appears when one of the conic sections in the image above (32) becomes a circle.

Figure 57. Given a conic section, one of the focal points and a circle with center in the focal point. We draw two lines through the focal point, and their diagonals with conic section and circle, will meet on the diagonal between circle and conic section.

Since we know the circle, we can find properties of the conic section.

157. First we look at the relationship of the parabola to the focal point, and will find the theorem:

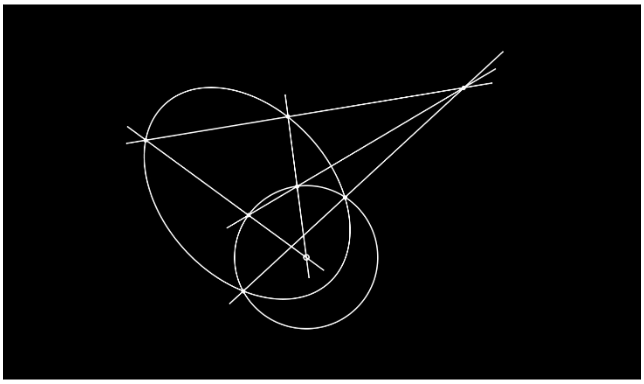

Figure 7.4: Focal point circle

Metric law 13. *From all points on a parabola, the distances to the focal point and a specific line, the control line, are the same.*

This is perhaps the most common phrase for parables.

158. To show the theorem, we let the conic section in the image above (57) be a parabola, and we let the circle centered at the focal point tangent to the parabola. One of the lines from the focal point g passes through the point of tangency and the focal point, while the other is free. One line meets the parabola at infinity, and the line from here to the second point p of the parabola becomes parallel to the first line. We draw the diagonal between the circle and the lines, and the common tangent between the circle and the parabola, and these meet at a point. We then have the image p a fig.(7.5). Here we see that the distance from the focal point B to the point P is the same as the distance to a line s. Here s is *the guiding line* of the parabola and the theorem above is shown.

159. We shall n a generalize this to a apply to the general relationship between the focal point of a conic section and the directrix. This is given by:

Metric law 14. *From any point on a conic section, the ratio between the distance to the focal point and the distance to a specific line, the directrix, is constant.*

This relationship is often used as an elementary definition of a conic section because by varying the constant, all conic sections are given. If the constant is less than 1, we have the ellipse, if it is equal to one we have the parabola, and if it is greater than one we have a hyperbola.

160. We make another modification to the circle-conic section theorem (57) for fö arrive at this relationship. We have an ellipse and a circle with a center at the focal point, but we have let the two lines merge into one that is a r

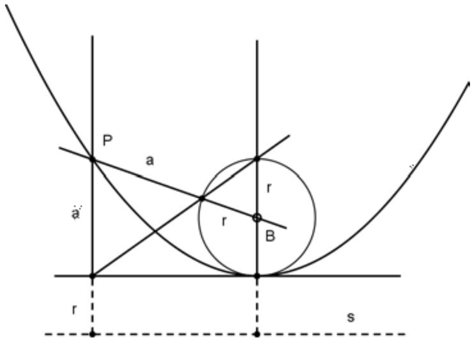

Figure 7.5: Focal point parabola

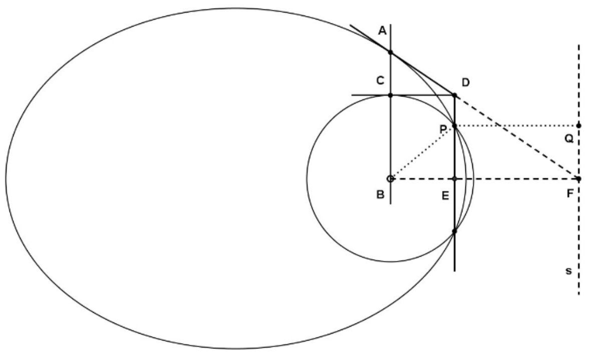

Figure 7.6: Ratio of ellipse

normally on the longest axis of the ellipse, as shown in the figure (7.6). Where the line meets the circle (A) and the ellipse (C) we add tangents, and these will meet on the diagonal between the circle and the ellipse in D. The line AD meets the axis in F, which lies on the normal s, the control line of the ellipse. We can see from the figure that BP is equal to BC which is equal to DE. Na, the relationship between DE and EF is equal to the relationship between AB and BF, which is constant. This means that the ratio between BP and PQ is also a constant, and by a varying the radius of the circle we have the metric ratio.

161. By seeing that AD is a tangent to the ellipse, we realize that s is the polar the focal point. We can therefore define:

Definition 6. The guiding line of a conic section is the polar of the focal point with respect to the conic section.

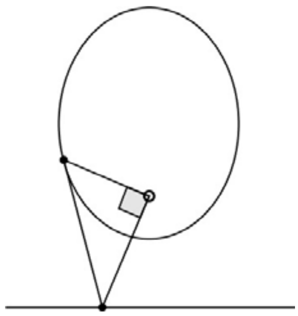

Figure 7.7: Right angle

162. By°a letting the circle decrease, the diagonal between the ellipse and the circle will p°and as the circle remains inside the ellipse, the diagonal will g°a through imaginary common points, and remain p°a on the outside of the two. This line can be found by°a drawing lines through the focal point, and finding diagonals between these and the circle and ellipse.

163. We will tn°make a further change, and allow the circle to become smaller and smaller so that it eventually coincides with the focal point. This is not immediately feasible because the intersections between the double line and the circle disappear. However, we realize that in the limiting case, this diagonal will become a bisector for the angle between the two lines through the focal point. This bisector will then meet the diagonal between the double line and the ellipse p°a the diagonal outside the ellipse. The diagonal is n°a become the guiding line of the ellipse.

Figure 58. Given an ellipse and a focal point, and a double line through the focal point. The diagonal between the double line and the bisector of the double line will then always meet on the same line, which is the guiding line of the ellipse.

164. This can be further condensed into a familiar phrase.

Figure 59. Given an ellipse, a focal point, and the guiding line in relation to this focal point. We draw a line through the focal point, and a normal to this at the focal point. This normal will then meet the tangents to the ellipse where it is intersected by the first line.

From these sentences we can also°find metric sentences.

165. We see from this that the definition of the focal point as the intersection of two tangents from the circle points agrees with other definitions of the conic sections. We will later see that other definitions based on the focal points can also be deduced quite directly from the images that arise.

Chapter 8

Absolute conic section

166. During the 19th century, projective geometry emerged, and eventually also an algebraic understanding of it. At the same time, another type of geometry emerged: the so-called non-Euclidean geometry developed by Laboshevki and Bolay. Even this geometry had its origins in problems with parallel lines. These directions found a kind of unity in that Cayley continued with the metric considerations made by Poncelet. While Poncelet made the connection between metric geometry and projective geometry by introducing the circle points, this was generalized by Arthur Cayley who not only linked metrics to the circle points, but to an arbitrary conic section; the so-called absolute conic section. This consideration also plays a major role in pure genometry. The imaginary enters here in a different way, and one gets an intuitive understanding of the nature of Euclidean geometry as a special form of geometry in general.

167. We shall primarily look at the purely geometrical considerations connected with this theme, and show how these arise naturally from the morphological movements. To some extent we shall also a look at a the metric; to a deepen the theme.

8.1 Salmon theorem and double tangent

168. The consideration will revolve around certain extensions of the theorems we have seen so far. We will form a synthesis of two and two of the three-cone carving theorems, but we can also take Brianchon's theorem as a starting point, which gives a more immediate access to this.

169. We consider that the transition from Pascal's theorem to Pappo's theorem occurred when the conic section in the configuration became two lines. We see n ap a the tangents in the hexagon in Brianchon's theorem, and ask n a whether these can be regarded as

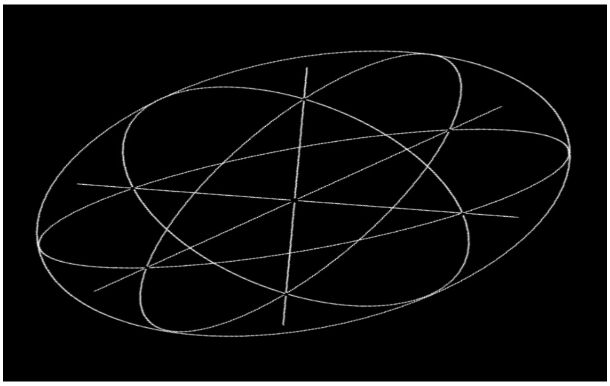

Figure 8.1: Solomon's theorem

degenerated cone cuts. This turns out to be the case, and we then have three cone cuts that double-tangent a fourth.

Figure 60. *Given a conic section, and three others that are tangent to this double between two and two of the conic sections, we find three diagonals, and these will meet at the same point.*

We call the theorem Salmon's theorem because it is expressed explicitly in his book "Conic sections".

170. Here we become aware of a new significant element; we are with direct relationships between the conic sections. So far, points or lines have been included as connecting elements in the various images, while here the cone sections themselves are linked by double touch or tangent. In order to avoid misunderstandings a, we will briefly say that two cone sections are tangent to each other when they are doubly tangent to each other.

171. We will call the central conic section the *primary conic section*, and the other three *secondary ones*. We do this not to emphasize any value for s a wide, but because they appear in this order.

172. Salmon's theorem is a synthesis of two three-conic section theorems. It one of the four central ones when the primary conic section collapses into a double point, and it becomes another when the primary section becomes a double line.

173. The dual of Salmon's theorem can also be formed by exchanging the intersection and diagonals with common tangents and common points. Then we have the theorem.

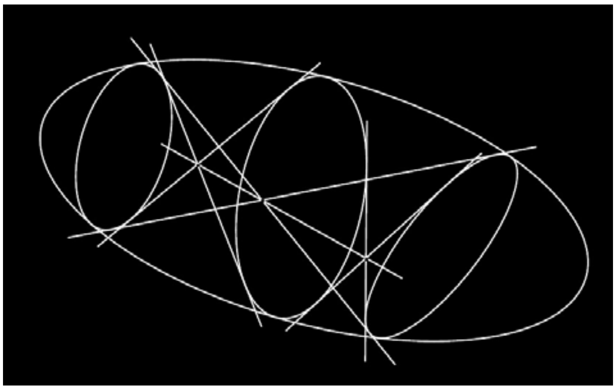

Figure 8.2: Dual Salmon

Figure 61. *Given a conic section, and three others that are tangent to it. We find three pairs of joint keys for two and two conic sections, and these will lie on the same line (Fig. 8.2).*

174. This theorem becomes the dual of the Salmon transformations.

175. If we leave the three conic sections in dual Salmon inside a fourth, but so that they do not touch each other, we will a Pascal's theorem n °a r the three conic sections become increasingly narrower, and finally collapse into three line segments that appear as three pairs of points. This process is dual to that which takes place n °a r Salmon's theorem g °a r over to Brinachon's theorem.

176. In the same way °that Salmon's theorem applied in a double-point variant and a double-line variant, we can form the dual cases here.

Figure 62. *Given three conic sections that are tangent to two lines. Then three diagonal points between them will lie on line.*

N °a r the outer conic section, double points are created °ar:

Figure 63. *Given three conic sections through three points. Then three diagonal points between two and two of the conic sections will lie on the same line.*

This theorem becomes Monge's theorem n °where the two points become circle points.

Figure 64. *Given three circles, and the outer common rod nets of two and two of the circles. Then the three common points of the tangents will lie on the same line.*

A number of other transformations can take place, and several of these are given as tasks under

177. Before we see how this leads to m, let's turn our attention to the diagonals again. When two conic sections , it is possible to draw a total of six diagonals. However, it is only two of these that are important: the two diagonals that remain when the two conic sections connected to the primary become two pairs of lines. Six such diagonals are formed, and they will meet three by three in four points. In each image, we will only be dealing with three diagonals a.

178. We thus have two main appearances of Salmon; in one case the secondary ones surround it, in the other they lie within it. We will soon see that each of these expressions can be differentiated into two different images.

179. Salmon can also be metamorphosed in that one or more of the secondary cone sections appear specially. They can degenerate into one or two pairs of lines, and each of these will give rise to distinct images. When the conic sections become two pairs of lines, we have, for example, a self-dual theorem that is a synthesis of Pascal's and Brianchon's.

Figure 65. Given two conic sections that are tangent to each other. On the inner conic section we put on each side a pair of tangents, and we find two pairs of points on the outer one furthest from the inner one. Each pair of lines that meet also form their own points, and we draw the line between them. Two diagonals between the pairs of points will then meet on this line.

When the inner conic section becomes a pair of points, we have Pascal, and when the outer section becomes a pair of lines, we have Brianchon.

8.2 Absolute conic section

180. We shall n ar determine some properties of the conic sections from the primary conic section that we n a see p a as the absolute. This f ar n a role as the determinant, and it will thus replace b ade the line at infinity and the circle points. The conic sections that are tangent to these are regarded as circles in such a system, and the properties of these circles are determined by this. We shall see how the center of the circle is determined, and the properties associated with this.

181. Significant for these considerations are the transitions that result from the secondary conic sections either coinciding with the primary or with each other. This gives rise to ar generalization of common diagonal sets, and these play a major role.

182. We first see what happens when one of the secondaries coincides with the primary. We then keep the tangent points of this fixed, and let it s a end.

so that it approaches the primary. We then see that the diagonals between this and the other secondary gradually approach the common diagonals between these and the primary, and when the cone intersections coincide, the diagonals merge with these.

Figure 66. *Given a primary cone seat and two secondaries. Then the common diagonals between the primary and the secondary will meet on a diagonal between them.*

This is analogous to what can be found at where the primary conic section is line pairs.

183. When two of the primary ones coincide, the image has a slightly different appearance.

Figure 67. *Given a primary cone seat and two secondaries. Then the diagonals between the two secondaries will meet on a diagonal between a primary and a secondary.*

184. When we make the same movements with the dual Salmon, the dual images appear. The images can arise through dualization, or dual morphological processes. We will not go through this in detail, but outline the sentences that arise. The elements that appear here will shed light on the properties of circles in an absolute conic section.

185. We first make the movement where a secondary conic section coincides with the primary one. Then we have to start from a modification of the image in the figure. We still have three cone sections inside a primary one, but we find the outer common points between the middle one and the other two, and the inner ones between these two. These then lie on a line. We then allow the middle one to grow so that it eventually coincides with the primary. Then the common tangents between this and the two will be the tangents where the two are tangent to the primary:

Figure 68. *Given a conic section and two others that are tangent to it internally, so that they do not intersect. We find the common tangents between the primary conic section and the other two, and these meet at two points. The line through these will go through the inner common point between the cone sections.*

186. Here we have a fundamental relationship in a space formed by an absolute conic section. Namely, we say that the common points between the primary and the other conic sections are their centers with respect to the absolute conic section.

Definition 7. *The common point between a conic section and an absolute conic section, we call the center of the conic section with regard to the absolute.*

The theorem above then says that the line between the centers of two circles "will go through their inner common point. This is, of course, an elementary theorem when it comes to circles.

187. Now we consider the dual element between the absolute and the other conic sections, namely the diggonals in the tangent points. These are in a way the dual centers of the conic sections, and we call them *centrics.*

Definition 8. *The diagonal between a conic section and an absolute conic section, we call the zenith of the conic section with respect to the absolute.*

One of the sentences above then expresses: Given two conic sections in an absolute. The two centers then meet on the diagonal between the cone sections.

8.3 Imaginary tangent

188. In contrast to the presentation we have seen so far where we went into the concepts of imaginary points and lines, we will see what we call imaginary tangent between two conic sections. We also start here with Salmon's theorem, and let two of the conic sections merge into one. Then the diagonals between them will merge into one, and go through the tangent points. This was a movement we have done in the past when dealing with pole and polar, now we do the same movement where we have not had degeneration of any conic sections. We obtain the theorem:

Figure 69. *Given a conic section and two others that are tangent to it externally. Then the diagonals between the two conic sections will m ates p a the common diagonal of the two conic sections.*

Due to the symmetry, we realize that both joint diagonals will meet at the same point as the diagonals.

189. This theorem can be transformed into a new one by letting one of the outer conic sections become a line pair.

Figure 70. *Given two conic sections that double tangent each other, and two tangents to one of them intersect the other. Then we can find diagonals between the double line and the conic section, which will meet at the common pole of the conic sections.*

By this theorem, we can construct a conic section n ar et is given, and more specifically if a circle is given. The process is the same as the envelope in Brianchon's theorem.

190. By changing the position of the common pole, the location of the enveloped conic section will change. We let the common pole approach the periphery of the enveloping conic section, and as it tangencies, we will have four-point tangency between this and the conic section being formed. N ar s if the pole remains outside the generating conic section, the enveloped conic section will not

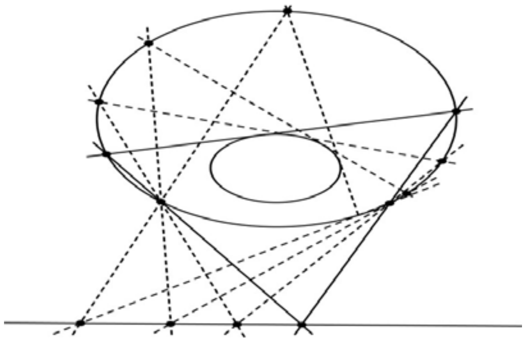

Figure 8.3: Imaginary tangent

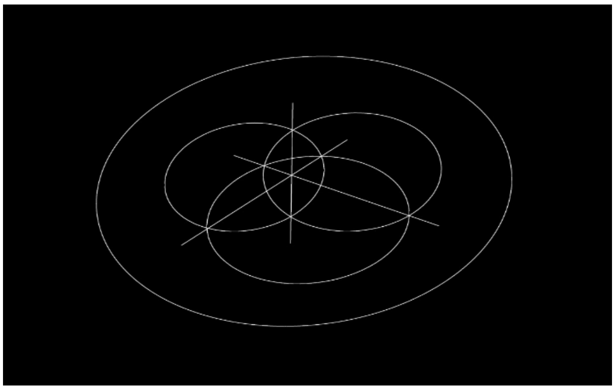

Figure 8.4: Ellipse room

touch this real, but n°a arises°ar imaginary tangent between them. The trap coil is thus°a located on°the outside, while the common pole is located inside both cone sections.

191. If we place two other conic sections inside this one, and find three diagonals between them, they will meet at the same point. The picture we have here is very similar to the situation with three circles, and it turns out that all conic sections that lie inside another p°a this m°aten form a set that is equivalent to the set of circles. Many of the properties between the circles can be found here; for example, we can apply Gergonne's construction and find ellipses in this set that are tangent to a given tree. However, not all elements are the same. For example, two diameters through the center of such a conic section will not give two parallel lines through the intersections. These will meet a°the common polar of the Absolute and the current conic section. We thus have

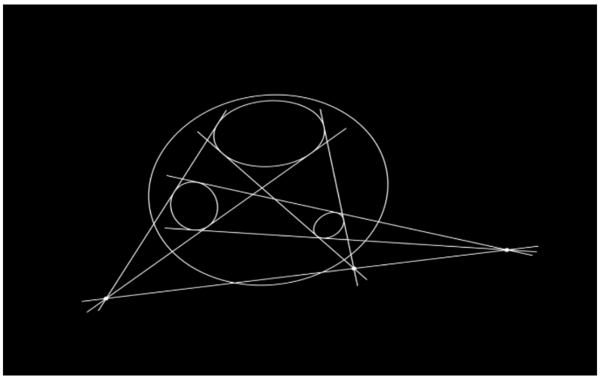

Figure 8.5: General Monge's theorem

a non-Euclidean circle geometry, and we can transfer all circle theorems to such a situation. By Cayley's metric, °a metric relationship can also be treated.

8.4 Formation of Euclid's geometry

192. We will n°a see p°a various what happens n° when the absolute varies. This was one of Felix Klein's ideas, °a see how the Caylik metric went together with the non-Euclidean metric in different contexts. Such transformations are treated later by Yaglom.

193. We will n°a see p°a the case where the Absolute conic section is a circle, and what happens n° when this becomes infinitely large or infinitely small. In the first case, we make it so that the Absolute lies around the other conic sections. We let s°a one of these conic sections remain in place so that its center lies together with the center of the absolute. This will then be concentric with this, and also °circle. We see this in the construction. If we allow °the absolute to grow outwards, while the other conic sections remain at rest, these will all gradually come closer and closer to the center of the circle, and as the absolute becomes infinitely large, all the other circles will become concentric with it, and thus circles. This is how °and °all circles arise as the amount of kje- glensite tangent to a circle at infinity. From a purely geoemtric point of view, we can just as easily set this as a condition as the circle points.

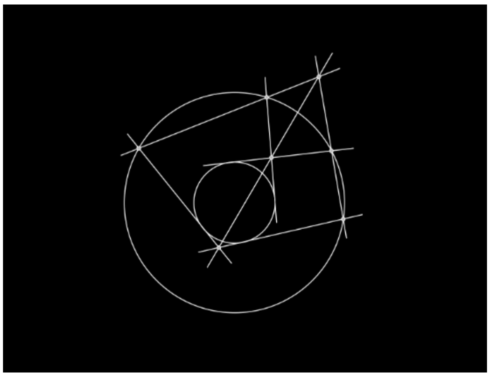

Figure 8.6: Two views together

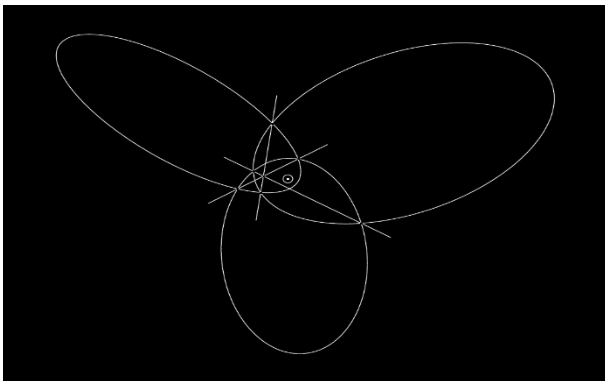

Figure 8.7: Focal point room

8.5 Formation of counter-Euclidean geometry

194. We g°a r n°a the other way, and let the absolute be a circle that is imaginarily tangent to conic sections, but lying inside them. N°a we let the circle decrease, and eventually let it collapse to a point. This circle will then be the focal point of all the other conic sections. The set of all conic sections that have a common focal point thus forms a geometry that is the opposite of Euclid's geometry.

195. We will see that it is the focal point that has been formed by°a see how we arrive at the image in the previous chapter from which we defined the focal point. We start with a circle that lies inside two conic sections and is imaginatively tangent to them. In addition, we have a hyperbola that is doubly tangent to the circle in real terms. Since the three conic sections all touch the circle twice, they will have three diago-

nals that meet at the same point. Again, we let the circle get smaller and smaller, and finally become a point. The conic sections that touch imaginatively will then have the point as their focal point, while the hyperbola will have become increasingly pointed and finally become two lines. The image that has been created is the one we used as a starting point when we defined the focal point in purely geoemtric terms.

196. It can also a happen that two of the conic sections become line pairs. It is then not so easy to determine the diagonal between them. If we allow the conic sections to become line pairs already while the focal point is a circle, we can see something that makes sense. We then place the two pairs of lines on the circle so that they form a rhombus. The diagonals in the rhombus are the bisecting angles between the lines. The two pairs of lines intersect the third conic section at two pairs of points, and the diagonals through these will then meet at the bisector. W h e n the circle becomes the focal point, we have the theorem:

Figure 71. *Given a conic section, one of the focal points and two pairs of lines, with the same angle between them, through the focal point. The pairs of lines form diagonals with the conic section, and these meet at the bisector between the lines.*

197. N f the line pairs collapse into single lines, we have:

Figure 72. *Given a conic section, one of the focal points and two lines through the focal point. The tangents where the lines meet the conic section then meet at the bisector between the lines.*

This theorem can be used to construct a conic section inscribed in a triangle.

198. Another variant of the double pair of lines is where one line from each pair coincides.

Figure 73. *Given a conic section, one of the focal points and two lines normal to each other through the focal point. From the points of intersection between the conic section and one of the lines, we draw new lines to a point på the normal. These will then intersect the conic section so that lines from here to the focal point so that the angle between them is halved by the normal.*

199. A final variant that occurs in particular we have når puntet på normale går to infinity.

Figure 74. *Given a conic section, a focal point and a line through it. Where the line intersects the conic section we erect normals, and where these intersect we draw lines to the focal point. The two triangles that nå are formed are equilateral.*

200. Another variation of this can be found at where both pairs of lines coincide.

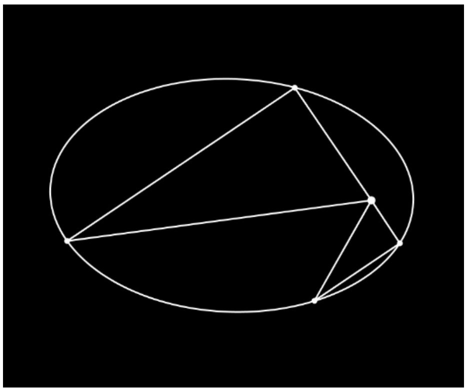

Figure 8.8: Equilateral triangles

Figure 75. Given a conic section, the focal point and a line through the focal point. Where the line intersects the conic section, we add tangents, and these meet at a normal from the center.

201. These three also meet on the directrix of the conic section. To see this, we return to the variant where we have a conic section, a circle and two lines through the focal point. Then the diagonals between the circle and conic section meet, and their diagonals meet the pair of lines in the same point. Now we let the circle get smaller and smaller so that it no longer intersects the conic section. Their common diagonal goes outside both, and when the circle goes together with the focal point, the diagonal becomes the control line. At the same time, the diagonal between the pair of lines becomes a bisector for the angle between the lines. Thus we have:

Figure 76. Given a conic section, a focal point and its guide lines. Through the focal point we draw two lines, and a diagonal between the pair of lines and the conic section will meet a bisector of the angle of the directrix.

From this theorem we can also find the ratio theorem. This is based on the angle bisector theorem.

202. When the two lines merge into one, we have:

Figure 77. Given a conic section, a focal point and its guide lines. We draw a line through the focal point, and where this meets the conic section we add a tangent. This will meet the normal of the line through the focal point on the directrix.

We see that this is half of the theorem above, but here we also have a directrix.

203. We can find many theorems in this geometry that correspond to the circle theorems, including the three-circle theorem. From one point of view, the planetary system is such a counter-space system where the planetary orbits are imaginarily tangent to the sun. It is interesting that the Copernican turn°a was°a g°ar away from systems with circles, to systems with conic sections through the same focal point.

8.6 Galilean geometry

204. With the image of all conic sections with a common focal point, we can make one more movement. We can let the focal point g°a to infinity. Then all the conic sections become parabolas in the same direction, and this collection also°a forms a three-dimensional geoemtry; Galilean geometry, where also°a relationships from circular geoemtry can be found.[1]

Figure 78. Given three parabolas with axes in the same direction. Then three common diagonals between the parabolas will meet at the same point.

205. It can be shown how all the properties of circular geometry are found in transformed form in Galilean geometry. However, it is beyond°a r frame a°g°a further into p°a these geometries; here we should only show how images from the morphological geometry are found. The main point from v°art's point of view is that, morphologically speaking, the conic sections that set the framework for the others belong to the overall picture. From this point of view, the elements in a room belong to the whole they°are part of. The relationship between the various elements will become even clearer in the next chapter.

Metrics and absolute conic section

206. The metric in can also°be followed from the absolute conic section to the various specializations. Some considerations about radii and their relationship to each other will be made here.

207. The diameter of a conic section with respect to°an absolute conic section is found a°is the double head that it forms with a line through the center.

Definition 9. Given an absolute conic section A and a conic section C in it. A line through the center of C intersects A in P and Q and C in S and T. We then define the diameter of C with respect to p a° A to a° be a double ratio formed between the points.

$$d=\frac{PQ \cdot TS}{QT \cdot SP} \qquad (8.1)$$

[1] In his book, Yaglom has made comparisons between circular geometry and Galilean geometry, and also°a what he calls hyperbolic geometry.

When one operates within the general space one must make certain logarithmic considerations. We look at an infinity relation, and then this disappears.

208. When we allow the absolute conic section to grow so that it eventually becomes the infinite circle, then the conic section will become a circle. The double ratio will also grow and go towards infinity. However, if we look at two circles next to each other, we see that the sizes that go to infinity are the same for both circles, so that the ratio between the two diameters is the same as the ratio between the ordinary diameters.

209. It is slightly different if we choose the definition above as diameter, or if we choose the inverse double ratio. In that case, the ratio of the diameters becomes the ratio of the inverse radii. And it is the case that both these quantities occur in geometric contexts. We look at two, which we also shall consider when the circle becomes a conic section.

Chapter 9 conic section type

By the synthetic comparative process we have now °brought it so °far that we can link theorems together in two comprehensive theorems. In Salmon's theorem and the dual to this lies as possibility all the special theorems. However, these theorems are not already °a understood °a as basic images. Even though there is a great deal of symmetry in the images, they are still °a special in their appearance; among other things in that they appear dually. We shall now °a look at°synthesis that also °a allows these theorems to appear as one; this will then be the original image or type of the entire field °of phenomena that has been clarified. The type will primarily °be with a symmetrical structure in °a similar way °as Desargues' theorem. It presupposes a transition of a certain kind; and this is precisely a transition that has to do with Desargues' theorem a.°

9.1 Formation of Desargues theorem

210. We have not yet ° seen Desargues emerge morphologically from other sentences, but here we will look at °this transition. The fact that a pure structure of points and lines can emerge is evident in the transition to Pappo's theorem. This happened when conic sections became point pairs and line pairs. In order for Desargues' theorem to emerge, a process of a somewhat different nature is required that we have not yet °studied. The transition exists in °two dual ways.

211. We begin with Salmon's theorem, and connect n °a hyperbolas to the primary conic section. We then start with an elongated ellipse, and let three hyperbolas double tangent to it externally so that one part is tangent to °the upper side, and the other to °the lower side. We also let the hyperbolae be narrow so that they intersect on °the upper side. We draw diagonals between two and two of the hyperbolas, and these three will meet at the same point. This makes sense from Salmon's theorem, only we have n °a with hyperbolas instead of ellipses a°

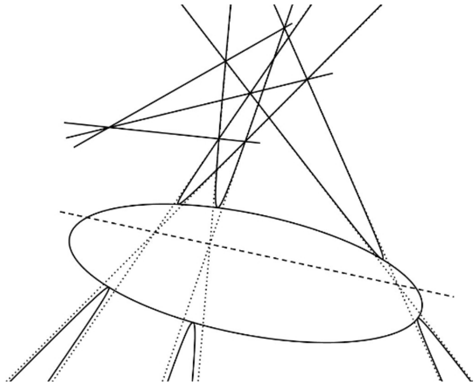

Figure 9.1: Transition to Desargues

Do. We let n°a the elongated conic section become increasingly narrow. This causes the hyperbolas to become increasingly pointed, and as the ellipse collapses into a line, the three hyperbolas will become three pairs of lines that intersect on°this line. W ° h e n we study the configuration that is created°more closely, we will see that it is Desargues' theorem.

212. Thus, °a Desargues theorem is also determined as a consequence of Salmon's theorem. We do not °a consider that Desargues theorem is self-dual, there is no difference in the structure with respect to °a points and lines. It is therefore not ° surprising that also °a the dual theorem gives Desargues' configuration. To °arrive at this we do a dual operation.

213. We start here with an ellipse and three other ellipses that are tangent to it externally. On °each pair of conic intersections, we add external joint tangents, and they will meet each other in three points located on °the same line. This is a variant of the dual Salmon theorem. We make the n °of the inner ellipse smaller and smaller, and let the outer ones follow this; however, so that the length p °of the outer ones is approximately the same. The outer conic sections then become increasingly narrower, and as the inner ellipse merges °intoapoint,theouter ellipses will become three line segments through the point. The formation that is created °is again Desargues' configuration.

9.2 The general type

214. We thus see that Desargues' theorem is a specialization of both dual images. By °looking at the formation process, we will be able to find a synthesis of the two. In one case, we had four conic sections, and three lines through a point.

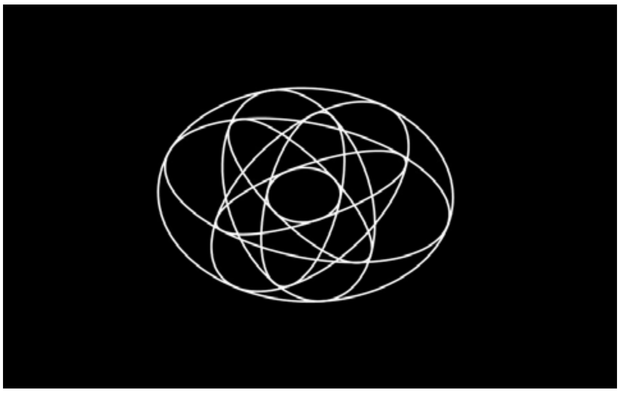

Figure 9.2: Image på type

Of the four conic sections, three pairs of lines were formed with common points on a line. In the second case, we had four conic sections and three pairs of lines with common points on a line, forming three lines through a point. We see a three points through a line can be formed by four conic sections, and like a three pairs of lines with common points on line. It is then obvious to ask whether the three lines through the same point in the Salmon configuration are four degenerate conic sections. Similarly, we can ask whether the three pairs of lines, and the line through the common points also a are four degenerate conic sections.

215. It turns out that such a synthesis is possible. The three lines through a point in the Salmon configuration can be resolved into three conic sections that are tangent to a fourth. The three conic sections are each tangent to the two they were previously diagonal to. In the dual case, the three pairs of lines resolve to three conic sections that are also tangent to two and two others, and the line resolves to a conic section that is tangent to these. We thus have a total of eight conic sections in the configuration, and can write this as a theorem.

Figure 79. *Given a conic section A, and three others B, C and D that are doubly tangent to it. Furthermore, three new conic sections E, F and G are tangent to two and two of B, C and D. Then there is a conic section H that is doubly tangent to E, F and G.*

To have a direction in the description, we can also a call the first conic section the primary, the next three the secondary, and then a the next three tertiary, and finally the quaternary.

216. We have thus arrived at the fundamental theorem in this presentation, or what we can call the conic section type. It consists only of conic sections that double-tangent each other in a particular structure. All the theorems we have seen so far find their root in this one image. The image is completely symmetrical,

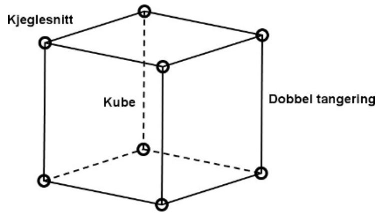

Figure 9.3: Structure of the type

we are dealing with only conic sections, and each of the conic sections doubles three others. Thus we have arrived at a picture of a similar nature to Goethe's Primordial Plant, where he considers that the plant best consists of transformed leaves only. In the same way, the various theorems within conic section geometry, seen in this context, are to be regarded as images where all elements are transformed conic sections.

217. The theorem has a specific structure that also a shows that we are dealing with something typical a. The structure of the conic section theorem is the cube. The conic sections are represented by the vertices, the connections between the conic sections are represented by the sides of the octahedron. The entire conic section structure is represented by the cube. We can also say that the cube is the graph of the type and that the conic sections are the basic elements.

9.3 Definition or evidence

218. The course of our investigations has been a g based on specific theorems, and through certain synthetic processes we have found increasingly general correlations. As the general ones have emerged, we have investigated the consequences of these sentences, and in this way we have substantiated has been found. When we come to the type that represents an end point in the investigations, we stop and ask ourselves how this should be understood.

219. Originally, conic sections were defined as sections of a cone, and by looking at them, other phenomena could be found associated with them. However, we have seen that many sentences can be used as definitions. Thus, we see that Pascal and Brianchon both give an implicit definition, and we have achieved add that also a relation theorem occurs among others. We now ask a: can the type be used to define the ellipse?

220. This question is fully legitimate for several reasons. One reason is this. Which definition of conic section should we put first. If we have given one definition, others can be derived from it, and if we choose others, things can be reversed. N ar we have seen that all these definitions from this point of view are special cases of the type, then it can be considered to put this one first.

221. We can also consider how Desargues' theorem was first proved in various ways, but was eventually set as an axiom. This turned out to be the theorem that underlies others, and not vice versa. In the same way, the type turns out to be a symmetrical relation that is the cause of many others.

222. It is possible to prove the theorem and its variants algebraically, but with this we go beyond the purely visual geometric realm and use techniques. As one can be in the purely visual ideal geometric n ar in the case of a line point, so one could justify the conic section theorem as s adan.

223. There are certainly several reasons for putting type first, but then we have to ask: is this possible? What provisions do we need to make to have a valid justification for this? We a therefore need to look at a the basic movements one more time, and see what kind it depends on a .

9.4 conic section based on type

224. The first thing we say in terms of definition is that we have a curve that is determined by p a one side becoming a pair of points, and p a the other side becoming a pair of lines. P a the other side, these curves form a cube structure.

- Conic sections are curves that on one side degenerate into two points, and on other side into two lines.
- The curves touch each other twice and have five degrees of freedom.
- The curves are entered in a hexahedron structure

225. It turns out that we're only getting some way with this, but we're not getting to all the configurations. We can only realize this by counting. For example, if four conic sections become pairs of points, and four become pairs of lines, then we only have eight points and eight lines. In Pappos we have nine of each, and in Desargues ten of each. There is something extra here, and we will eventually look at this. First, however, we will see how far we have come with the above definitions.

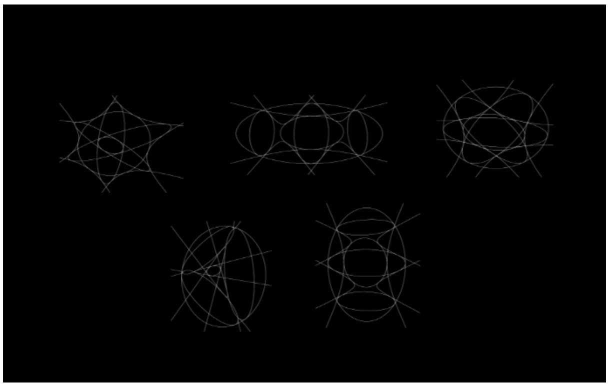

Figure 9.4: Variants of the main theorem

226. A particular difficulty arises that is not easily overcome. N ° a r we follow the process: We start with a conic section, find three secondaries, then three tertiaries, then there is exactly one quaternary tangent to these three. The problem is n°a that we cannot abstractly say where this lies. We have to ap °a sort of point, we cannot say in words how it will lie.

227. We n ° a r with the definitions above just to what is called Poncelet's theorem [1]. We start with a primary conic section, add three pairs of lines that touch this, between two and two lines we find pairs of points, and through the six points formed g ° a r a conic section. This can be expressed as:

Figure 80. *Given a conic section that is inscribed in two triangles. Then is a conic section that circumscribes the triangles.*

This can also be said as; given a triangle that circumscribes a conic section, and a conic section that circumscribes the triangle. Then there is a linear set of triangles between the two.

228. Here we see that the primary conic section is still a conic section, the three secondary ones are line pairs, the tertiary ones are point pairs, and the last one is an ordinary conic section again.

229. By just following the definitions we have made we will not get any further, there will always be two conic sections in the configuration, and for example Pascal's and Desargues theorems will not appear. We are going to see what transformation we need to put in addition to pure point and line transformation of the conic sections. We will then start from a particular theorem.

[1]This is called Poncelet's theorem and applies to all polygons; if a polygon lies between two conic sections, then there is an infinite number of conic sections between them.

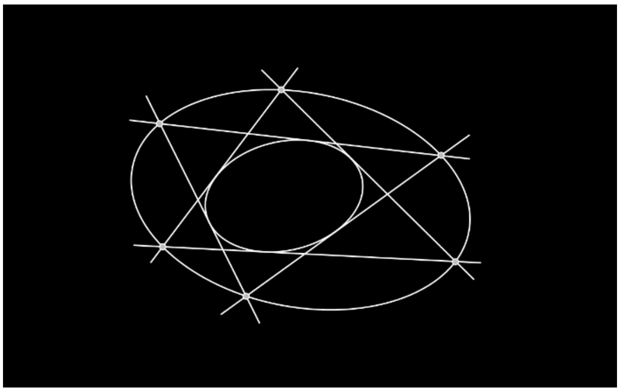

Figure 9.5: Poncelets rate

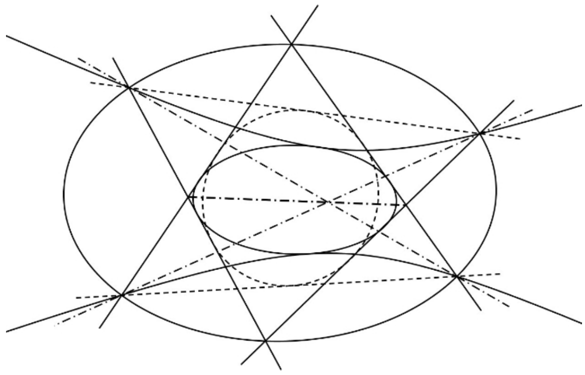

Figure 9.6: Transition between Poncelet and Pascal

230. In the formation of Poncelet's theorem, the three tertiary conic sections became line pairs. If we°move one of these line pairs, so that it becomes a hyperbola, then°an intermediate variant arises between Poncelet's theorem.

Figure 81. *Given a conic section and two lines through each of two points på per-iment. These form four new points på the conic section, and we let a conic section g° a through these four points. Then the conic section and the lines will inscribe a new conic section (Fig. 9.6).*

231. This theorem can nå be specialized in two directions. We see from the figure (Fig. 9.6) that n°ar the hyperbola straightens out and becomes a line pair, then the inscribed conic section will , and we f°ar Poncelet's theorem. However, if the hyperbola becomes more pointed, so that it eventually becomes a line pair that crosses the configuration, then the inscribed ellipse becomes increasingly narrow, and g°ar så over to å become a

line segment.

232. It turns out to be difficult to determine in the abstract when the first case occurs and when the second takes place. For now, we must say: *When transforming a conic section into two lines, the conic sections concerned may still be conic sections, or one of them may degenerate into a line segment where the two lines meet.* This must be added to the definitions in the introduction.

(Rest of chapter incomplete)

9.5 Coinciding conic section

233. One consideration that can be made before the conic sections degenerate into point pairs or line pairs is a see p a conic sections that coincide. We have seen p a that elements coincide in several contexts; n a we will try to a see this from the type.

234. The first type of coincidence occurs when two secondary conic sections touch the primary at *the same place*. Here we understand intuitively what the same place is. Other than a, we will not describe this using the elements point or lines, because these have not arisen yet a. We will follow the process for a to see what happens with other conic sections in connection with this.

235. We then follow what we might call the basic four-conic section relationship, a primary conic section, two secondaries, and a tertiary. When the points of contact between the primary and the two secondaries approach each other, we also see that two points of contact between the secondaries and the tertiary this point. In the case of coincidence, the four sites come together.

Axiom 3. *If two conic sections touch a third at the same place, then all conic sections touching these two will touch them at this place.*

236. We determine this purely from the conic sections. If n ar one of four coincides to form a line segment, the point will also a lie p a this place, so that conic sections that touch each other p a the same place, have a common point p a this place. The dual happens when we f have lines, then f the conic sections have a common tangent p at this point. We therefore have: Two conic sections touching each other have a common point and a common tangent p at the point of contact.

237. The basic coinciding movements we can also a see on a. We can have two such movements. One is when two conic sections that are tangent to each coincide. Here it will happen that and the basic theorem that describes this we have at: Variant1

Figure 82. *Given a conic section, and two pairs of others that are tangent to it at two pairs of points. A conic section that touches these two will also be able to touch one that touches the other two.*

238. We have a special variant of this at

Figure 83. *Given a conic section and two that are tangent to it, and a diagonal between these. Also give two others that are tangent to the first in the same points as above. One of these intersects the diagonal at two points. If the second goes through one of these points, it also goes through the other.*

239. A variant that can be used to find tangential conic sections is given by

Figure 84. *Given a conic section, and two others that are tangent to it. Then there will be a conic section through the four tangent points and two of the intersection points.*

240. The second type of coincidence is where two conic sections that are both tangent to a third coincide. In the last case, we have an intermediate variant where the cone intersections touch at the same point. Here, their common cone sections will also touch at the same points, so that we have a total of four cone sections at the same location.

Figure 85. *Given three conic sections tangent to a fourth at the same two points, and another tangent to this but cutting across the other three. Two conic sections formed between two and two of these will, together with the fourth, tangent another.*

9.6 Formation of lines and points

241. We set nå up a modification of the theorem to å see på the movement we are missing. We then let one pair of lines be conic sections, and otherwise we have the same.

Figure 86. *Given a conic section, two pairs of lines and a conic section touching it, between these three pairs of points. Then there is a conic section through the points.*

242. We see that if the secondary conic section becomes a pair of lines, ar Poncelet's theorem arises. If we instead let the secondary conic section be a hyperbola that we make increasingly pointed, the conic section it touches will become increasingly narrow. As one conic section becomes two lines, the other will become a line through the intersections of the two pairs of lines, and also a through the intersection of the last pair of lines. The image that nå is created att is Pascal's theorem.

243. This transition s a we also a in connection with Desargues theorem, and it is not embedded in the definitions we have made s a so far. How should we describe this transition? Before we finally get into this, we will look at some other transitions that do not fall under our definitions so far.

244. We then start with

245. With this we do two things. We are in the developmental method, and see how the type applies in different situations. The second is that we also follow a scientific method. When one generalizes or introduces a new hypothesis, it should lead to more things being explained. Even if this demand on science is dogmatically put forward without ideal, but only practical justification, it is a path to follow. Only what is fruitful is true. We will then in the next chapters go a into a consequences of the main theorem, and also a examine the movements that can be made ideally. In the first chapter we make general movements, in the next chapters we seek to investigate different areas of the conic section theory as it exists.

9.7 View and spr a k

246. Once we have been able to put the various transitions into words, we can stay completely within the language to explain the various images. The language is freed from the view, we no longer need to point at the image to understand what is being said. It all becomes tautologies, but at the same time the language mirrors the perception so that it becomes conscious.

247. The type we have in mind is similar in that we say: Given a primary conic section, three secondaries tangent to this, three tertiaries tangent to two and two of these. Then there is a quaternary that is tangent to these. This statement applies s a all the time, but what we mean by conic section in different cases varies. Sometimes the primary is an ellipse, sometimes it is a pair of points, because a pair of points is a conic section, and so with the other elements.

248. We can see how Poncelet's theorem appears: We have the premises. A conic section that doubles another conic section can behave as a pair of points lying on the conic section.

249. By a seeing p a how we are in the actual or logical, we can see the extent to which we are still stuck in the view. We thus free the actual from the view. (Hilbert's system of geometry has been tried in logical machinery p a p a computer, and it turns out that he by no means only stuck to the logical, but constantly used geometrical views).

250. Morphological geometry thus liberates thinking on two levels. First, one is led up into the pure view where one frees oneself from the sensuous as

it is given to us with parallel lines, right angles and circles. Then we free the logic from the visible, so that the work is no longer bound to the visible, but mirrors it.

251. This type of liberation is also sought through algebra. Then you have an abstract level a where things weave together logically. However, a algebra must also a be said a to be subordinate to the logical in s a respect. (Spr aket, Logic and perception also a belong together)

9.8 Basic movements

252. Now that we have identified the symmetric type, it is appropriate to take a closer look at the movements we make to develop the various theorems. These movements have been more or less pronounced, and we shall now a clarify some key points. Pictorially, the transformation depends on the changes we can make to the conic sections and their positions in relation to each other. In addition, there are certain underlying transformation motives, the motives for changing these as we do.

253. We again draw attention to the fact that conic sections degenerate in one direction into two points, and in the other direction into two lines. The conic section can thus appear in two polar forms, and from a morphological point of view this is the reason for the principle of duality, among other things. Every theorem that we form morphologically, we can form dually by following a dual morphological path. What appears as a principle can be followed in real terms as a development process. Yet we cannot escape the mystery that the conic section has these great possibilities in it, that the fundamental polarity of the world shows itself so a pregnant in geometry.

254. Polar movements do not only occur within the image. In all acts of morphological formation we make inner polar movements of various kinds.

255. The first occurs when we think of the figure itself. We then set the first conic section, and to these we add others. To these we add yet others, until we end up with the final result. In this way, we set a direction that is not inherent in the archetype itself, but in order for something to emerge, we have to begin, develop this, and end. We have with a formation in time s a and say in order for something coherent to appear in space.

256. The next movement we make is a polarization in the morphological process. To change the whole picture, we fix some elements, while others have to move. We can't change all of them at once, because then everything becomes fluid and unchangeable. This process is similar to those we see in the organic world when an organism forms a bone structure on one side and blood circulation on the other.

other, or where in the plant we have the differentiation between stem and juices. We have recently seen such movements in the emergence of Desargues' theorem p˚a different m˚ater.

257. Another differentiation is that between the moving elements, there is one element that we set as the driving force, the one that leads the way, and the other follows it. This differentiation can be more or less obvious, but in the case of constructions it is clearly . Here we have an element that moves in one direction, or along a circle. Other elements follow p˚a, and a resulting element constructs.

258. Finally, conic sections that are of a higher order than lines and points are specialized to these. They die in˚a way into a lower world. Through movement, a row of the lower elements is formed, and the higher conic sections appear. We are then dealing with an incarnation of something˚, and this is made possible by the fact that something of the same kind has˚, so to speak, sacrificed itself.

259. In our thinking, we thus find several aspects of what precedes˚ar as driving elements in the organic world. W˚here the activity of the self is in the picture, we find it as life out there in the world.

9.9 Basic metamorphoses of the type

260. In the most general theorems we considered, we were talking about up to four conic sections in the images. Now˚we have˚to do with as many as˚eight conic sections weaving together. This does not make it easy˚to get an overview of the various movements, so˚certain systematic approach to the investigations is˚The first thing we need to˚a see on˚a is how different images are expressed in concrete terms.

Chapter 10

Circular seals

N°ar the original image, or type is found, then we have n°add a m°al in a certain condition. We have gained a view of the various images and theorems, which are all held together by one idea. But still°the type is not exhausted of possibilities, and there are n°a two paths for further consideration. One is still the synthetic method, the comparative method, where we see if we can find syntheses and higher laws. This endeavor is becoming clearer than in the past, when reason was the guiding°force. Now°we have a type that a current image possibly°fits into. We can then°a seek°a to resolve the various elements in the image into conic sections. (The possibility is always present that it does not°ar, that other elements mu°a be seen at°a.)

The other way is the movements based on the type. When°this is given, you can see what happens when°the different cone sections assume different shapes and positions. The imagination is used here, and you can see what°arises. From this process, completely new theorems that we have not seen before can come to light.

Goethe: If I were younger I would make a journey to India, not to°discover something new, but to°see this p°a my m°ate.

One aspect is also°that when°we find new forms from the imagination, these may turn out°to explain theorems that had not previously been seen in a morphological context. The author tried in vain°to find a satisfactory morphological solution t°two central conic section theorems, without success. One of these theorems was the definition of the conic sections as the geometric locus of constant sum. By free imagination, however, forms arose from which the solutions emerged i°a surprising way. We shall presently see the path leading to this theorem, which is part of an extensive class of theorems.

We then have the following in mind; what possibilities still exist as unfolding possibilities for the type? The other side of the question is, what conic section theorems of an elementary nature have we not yet°touched? We then recognize that we°not yet seen how constant sums come into being. Nor have we seen how we form

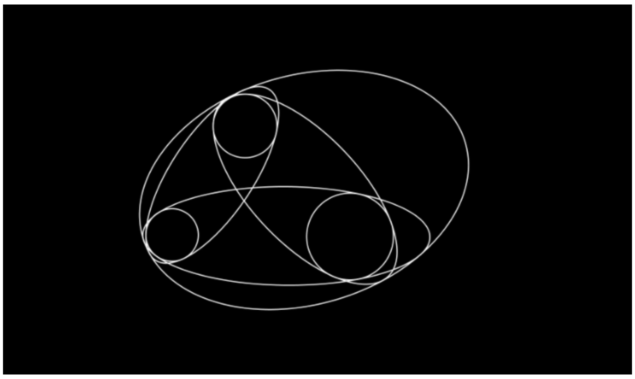

Figure 10.1: Three circles and four ellipses

conic section from circle. Both of these issues arise when a conic section becomes the circle points, so that several circles appear in the image.

10.1 The three circle theorem

261. We have already seen pa how three circles are formed by the primary conic section being the circle, or said pa another way, the infinitely large circle. Now we will extend this, and see p what happens when the primary conic sections are the circle points, but where we do not do anything special otherwise. Then the trisecondary conic sections will be circles, while the other four conic sections will still be general conic sections (fig. 10.1).

Figure 87. *Given three circles, and three conic sections that double tangent two and two of these. Then a fourth conic section will double the three conic sections.*

Although we have previously used the term the three-circle theorem, we will ana also use this term, and we will say the general three-circle theorem if there is any doubt.

262. We now have a class of theorems where three circles are always present. This means that it is possible to find metric relations since these are linked to circles. We also have a slightly larger overview here than when the primary conic section is two general points, we are familiar with the circle and hold it more easily in view.

263. Na arises from the general three-circle theorem the special three-circle theorem in that the three tertiary conic sections become diagonals and that the quaternary

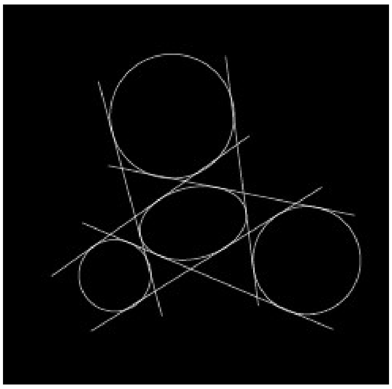

Figure 10.2: Inner joint keys

becomes their common point. As we have seen, Munge's theorem arises °when °the three conic sections become line pairs with three common points on°the same line.

264.

However, °when the tertiary conic sections become pairs of lines, we also have° another possibility; the three pairs of common diagonals can all lie between the circles. The intersection points between these will not be on°the same line, but they will touch the same conic section.

Figure 88. *Given three circles lying on°the outside of each other, and the six joint keys between the circles. The joint keys will then lie on°the same edge.*

The polarity between three lines through a point, and six lines on°the same conic section is also °apparent here.

265. A special variant of the above theorem appears when °the common conic section of the three pairs of lines g °changes to a pair of points. Then the tangents of the circles will meet three and three in two points. This can be written på another way°ate:

Figure 89. *There are three lines through each of two points g °. If two circles are inscribed by two and two pairs of lines, then a circle will also °be inscribed in the last two pairs.*

This variant does not exist n°ar the tertiary cone intersections become point pairs; this is because the circles have their other common points as circle points.

266. In the configuration, we have three lines crossing three others; så we have a kind of Pappos theorem for circles. We can further specialize this by°a letting lines fall

together. In one case, the lines from each point coincide and we får:

Figure 90. *Given two circles that are tangent to each other at the same point på a line, and two more points på lines. From the lines, we draw tangents to both circles, and the four tangents will then be tangent to the same circle.*

267. In the second case, two lines from the same point coincide; that is, they do not coincide so that the circle collapses, but they are separated so that the circle becomes tangent at one point.

Figure 91. *Given a triangle with a line from the vertex that divides the triangle into two parts. We inscribe in the three triangles a circle. Then a common tangent between the two smaller gå through the point of tangency of the larger på will be the common line.*

From this theorem we can realize several metric relationships.

10.2 Two focal points and constant sum

268. In this way, we can continue to move the elements in different ways. However, we keep in mind that we are looking for images where the focal points are included. We should arrive at an image where we can show the central theorem:

Metric law 15. *Given a conic section and the two focal points. From all points on the periphery, the sum of the distances to the focal points is the same.*

We are therefore looking for a geo-emphatic image where two focal points occur.

269. Immediately we find a theorem where three focal points are present. We imagine a three-circle formation with three circles and four conic sections. Så we let the three circles become smaller so that they eventually become imaginary tangents to the conic sections, and finally they become focal points. We then have the theorem:

Figure 92. *Given three conic sections where two and two have the same focal point. The conic sections will then be tangent to the same conic section.*

270. However, we will only have two focal points, and we can create an image in which two of the circles become focal points, while the third remains a circle. We then first create an image with three circles and three conic sections located so that the last conic section is not enclosing, but is an ellipse between the other three conic sections. The ellipse will become narrower, and when it becomes a pair of points, we have the image:

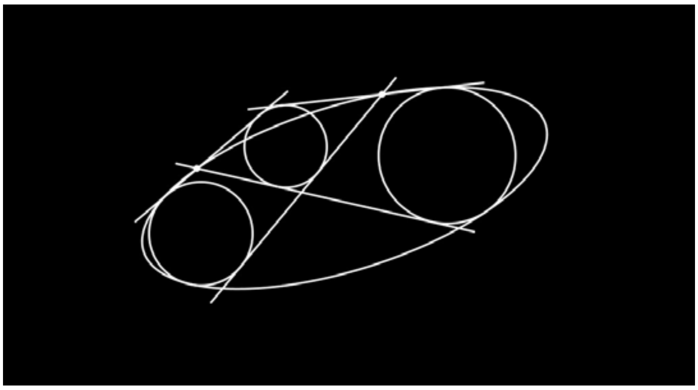

Figure 10.3: Constant sum regularity

Figure 93. *Given two conic sections that have a circle in common with a third conic section, and that meet at two points on the periphery of this. two conic sections will also have a common circle.*

271. We continue the process and let the two conic sections above become two pairs of lines. We then have:

Figure 94. *Given a conic section and two circles tangent to it. We add double tangents p to both circles so that one tangent from each meets at a point p on the periphery of the conic section. Then the four lines will inscribe a circle.*

Based on this theorem, we could arrive at what Jakob Steier says is a new definition of conic section. First, however, we'll see how it all turns out in a more special case.

272. What happens n°a is that the circles are imaginary tangents to the conic section. We let them s°a decrease so that they eventually become the focal points of the conic section. Then we have:

Figure 95. *Given a conic section, and a pair of lines through each focal point so that two and two meet at the periphery. Then the four lines will inscribe a circle.*

(Fig.10.4)

273. We have now°a arrived at a picture that in°a quite elementary way gives us the theorem for constant sum. By°a considering that the lengths of the tangents from a point to a circle are equal, we see from the figure (10.4) that the sum of the distances from the focal points to the periphery is equal to the sum of the tangent lengths of the circles.

274. We can make the same considerations before the circles become focal points (Fig. 94), and this is when Jacob Steiner's general statement arises.

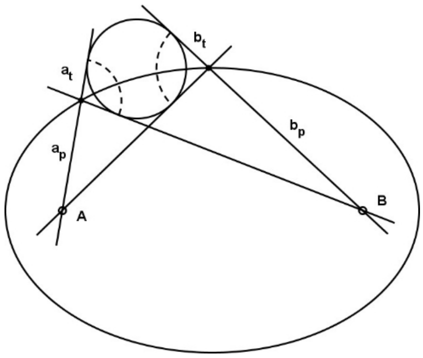

Figure 10.4: Constant sum regularity

Metric law 16. *All points lying such that the sum of the tangent sides of two given circles is constant lie on a conic section.*

By a varying, we also a arrive at similar sentences for parabola and hyperbola.

10.3 Hotspots as centers

275. We have previously defined the center of a circle as the pole of the line at an angle with respect to the circle. We then have, among other things, the property that when two circles are tangent to each other, then the line through the centers will g a r through the tangent point. In the counter-space image, the control lines are regarded as the centers of all conic sections with a common focal point, and when two conic sections are tangent to each other, the tangent at the point of tangency will meet the center lines at a common point.

276. If a it turns out, however, that n a r conic sections have a common focal point, then the other focal points will also a have the character of a being the center of the conic section in question. Among other things, the following theorem applies, which corresponds to the one above:

Figure 96. *Given two conic sections with a common focal point that are tangent to each other at a point. In this case, the tangent point will be in line with the other focal points.*

We will look at the formulation of the three-circle theorem that leads to these conditions.

277. The above theorem, among many others, ar n ar we make an opposite movement to the one we made for a theorem for constant sum. There we did a process where the quaternary conic section became two points. Now we let this become two lines instead, and when we reverse the order of elements that appear we can describe:

Figure 97. *Given three ellipses lying between two lines, so that two and two of the ellipses double the same circle. Then a the third pair of ellipses will also double tap a circle.*

From this theorem we can go in several directions, and we will show some images that arise before we turn to the specific issue of centers.

278. We can immediately let the one line g a to infinity. Then the three conic sections become parabolas, and we f a r the following simple theorem:

Figure 98. *Given three circles, and three parabolas that each double tangent two of these. Then the three parabolas will tangent the same line.*

This bidlet shows a fundamental connection between three circles and three parabolas.

279. We g a r back to the previous image, and change n a this by a letting one of the conic sections between the two lines become a line pair through the common point of the lines. Then there are only two conic sections left, and a rich contiguity between two conic sections is then given by:

Figure 99. *Given two conic sections between two lines that touch the same circle. We draw a line through the common point of the lines, and find s a circles each touching their own conic section, and which are tangent to the line. Then another common tangent to the circles will also a g a r through the vertex.*

This theorem is also the starting point for several correlations between two conic sections.

280. We can also a here let one line g a go to infinity, and the two conic sections become parabolas, while the lines become parallel.

281. If we let n a the two circles between the parallel lines get smaller and smaller, so that they become the focal points of the parabola, then we have the mysterious theorem:

Figure 100. *Given two parabolas that double tangent the same circle. Then a line through the two focal points will be parallel to a common tangent.*

282. The general variant of the above theorem is:

Figure 101. *Given two cone sections with a common focal point, the other two focal points, and two common tangents to the cone sections. Then the common point between the two will be in line with the focal points.*

By this we are close to the theorem we wanted to show. Also this theorem applies to circles; the common points of two circles are in line with the centers.

283. W h e n the two conic sections are instead tangent to each other, the common point between the tangents will be the tangential point, and we f a r thetheorem above. This f h a s a special form for the parabolas:

Figure 102. *Given two parabolas with parallel axes that are tangent to each other. A line through the focal points of the parabolas g a r then also a through the tangent point.*

Parabolas with a common focal point have common axes.

10.4 Equal sized cone snippet

284. We can also a see that the focal points can appear a as centers by ag a in a slightly different direction. For circles, we have the obvious theorem that i f the midpoint normal between the centers of two circles g passes through one of the intersections between them, then it will also ag a pass through the other intersection. This is not quite so obvious in the case of conic sections, but we have:

Figure 103. *Given two conic sections with a common focal point. If the center normal between the other focal points g passes through an intersection of the cone sections, then it g also p a s s e s through another.*

This theorem appears in a slightly different way. We look at to see what happens.

285. We f a r the theorem.

Figure 104. *Given two conic sections with a common focal point, and circles that are tangent, each of these is internal. If a diagonal between the two conic sections p a s s e s through one of the intersections between the circles, then it also p a s s e s through the other.*

Of course, we could also have the common focal point be a circle that doubles both conic sections.

286. At tangent we have:

Figure 105. *Given two conic sections with a common focal point, and circles tangent to each of these internally. If one circle is tangent to a diagonal between the two conic sections, then the other circle will be tangent to the diagonal at the same point.*

287. We have a special shade of this at w h e r e one conic section is two lines.

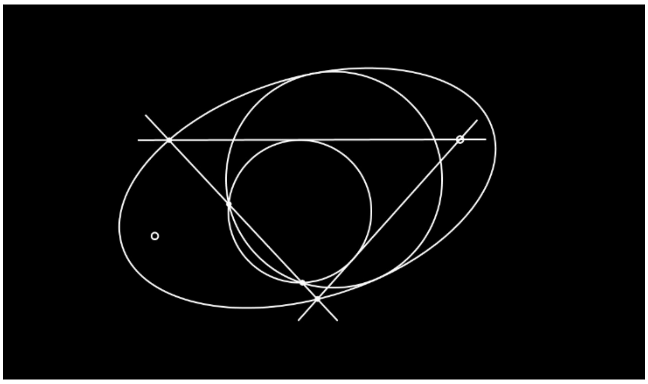

Figure 10.5: Circles with common diagonal

Figure 106. *Given a conic section, a focal point, and two lines through the focal point. We find the line between two of the intersection points, and the inscribed circle in the triangle will intersect the line at the same point as an inscribed circle.*

With this theorem, we can construct a circle that is tangent to a conic section and a given line.

288. When one of the conic sections becomes two lines f we have a special variant of it, and this f is a special expression for parabolas.

Figure 107. *Given a parabola, two lines parallel to the axis, and a line through the intersections of the parabola and the parallels. A circle tangent to the two parallels intersects the line at two points, and a circle double-tangent to the parabola will either g ar through both intersections or none.*

This theorem can be used to find circles that double the parabola.

289. We have a special variant for tangents.

Figure 108. *Given a parabola, two lines parallel to the axis, and a line through the intersections of the parabola and the parallels. A circle is tangent to the two parallels and the line, and a circle that is double tangent to the parabola and tangent to the line will be tangent to the other circle at the tangent point.*

10.5 Meeting between Euclidean and counter Euclidean

290. In the considerations we have just made, we have seen that we can make movements from circles to focal points. The reverse movements are also often found,

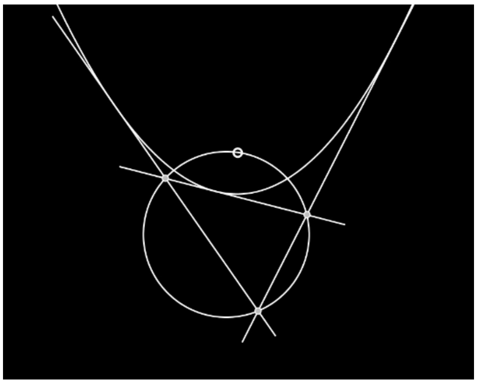

Figure 10.6: Parabola and focal point

n°f we have a relationship that involves a focal point, then we expect that this can be resolved into a double-acting circle. Surprisingly, this °a does not always turn out to be possible, and a theorem where this is not the case is a well-known theorem related to the parabola.

Figure 109. *Given a parabola and a triangle that circumscribes it. A circle that circumscribes the triangle will then gâ through the focal point of the parabola (Fig. 11.1).*

This theorem cannot be generalized so that the focal point becomes a circle that is double tangent to the parabola, and we will see the reason for this in a moment.

291. The above theorem follows from Poncelet's theorem (9.5), and the transition here is a classic example dfransition from two real points to the circle points. Poncelet's theorem states that when °two triangles are inscribed in a conic section, they also °rewrite a conic section. We let n°a two points p°a the same triangle be the circle points. This causes the circumscribed conic section to become a circle. The line through the circle points m°a is the line at infinity, and the inner conic section that is tangent to this line m°a is then a parabola. The two tangents through the circle points form the focal point of this parabola because they are tangents to it. The second triangle is not affected by all this and it rewrites the parabola. The circle rewrites it again, and g°ar also°a through the sixth point, which is the focal point of the parabola.

292. Here we have °two theorems that have the same structure, but which behave completely differently °when the circle points are taken into account. We realize n° a also°a the reason why the focal point in this case cannot be a circle; this is because the two lines that form the focal point come from different conic sections. We se°a how two opposite °sides of the hexagon came from a conic section; we mst°

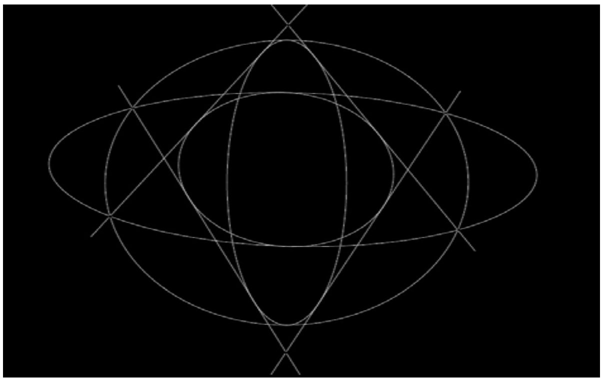

Figure 10.7: Relle points

therefore have a line from each triangle to form a conic section; here, two lines in the same triangle form the focal point.

293. This consideration leads us to ask for the most general configuration that lets a point from each of two conic sections form circle points, and a line from each of two conic sections form a focal point. It turns out to be a variant where we from the main theorem have let two conic sections become two pairs of points, and two others become two pairs of lines, so that the configuration is self-dual.

Figure 110. *Given a conic section that intersects another at four points, and a third that double tangents this. Through each of the four intersection points, we add a tangent to the double-tangent conic section. There is then a conic section that double tangents the other conic section and the four lines.*

We see here that two of the conic sections have four common points, and that the other two have four common lines.

294. We let n a two of the points originating from each conic section become the circle points. Both conic sections through these points then become circles, while the other two conic sections f are the common focal point of the two lines through the circle points. We can then set up the theorem:

Figure 111. *Given two conic sections with a common focal point, and circles that double each of these. These are laid out so that a common tangent to the conic sections g passes through an intersection point of the circles. The other common tangent g will then pass through the other intersection of the circles.*

295. We note that this configuration is completely dual, there are an equal number of points and lines in the configuration. Furthermore, it is self-polarized with respect to a focal point

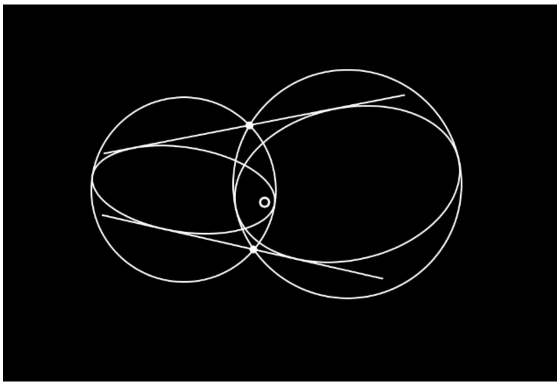

Figure 10.8: Two imaginary punts

and circle points. There are two circles associated with the circle points, and two conic sections associated with the two lines through the focal point. The theorem turns out to have a number of properties. We will make an initial observation that shows a specific property, before we look at more general variant.

296. We allow one of the cone sections to expand so that it eventually becomes a circle. This circle will then have the common focal point as its center. The circumscribed circle will coincide with this circle, and we have the following.

Figure 112. *Given a conic section, and a circle with the same center circumscribes this. Another circle has its center in one of the focal points of the conic section and a tangent to this circle where it intersects the first will also be tangent to the conic section.*

From the theorem f we immediately get the envelope direction, because we realize that the tangents must be perpendicular to lines from the focal point to the tangent point.

297. We also a come to the envelope law by a establishing another variant. We let n a one of the conic sections become narrower and narrower, and as it collapses it becomes a line segment, of which only the endpoints are significant. We then f a r the following conditions:

Figure 113. *Given a conic section, a circle that circumscribes it, and two tangents to it. A circle through the common point of the tangents, and through two of the intersections between these and the circle, will also ā through the focal point of the conic section.*

This is a simple and striking theorem. What is to note again is that the focal point in this case cannot be resolved into a circle because it is formed by two lines from separate conic sections.

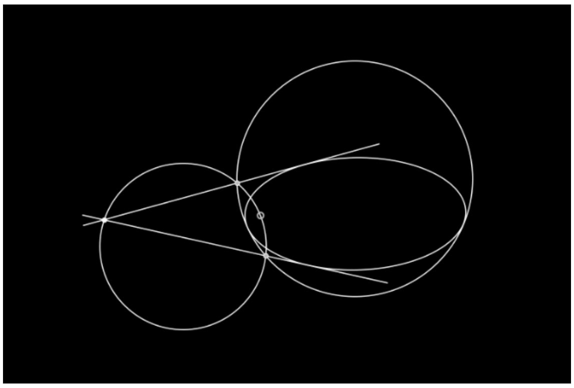

Figure 10.9: Circle through focal point

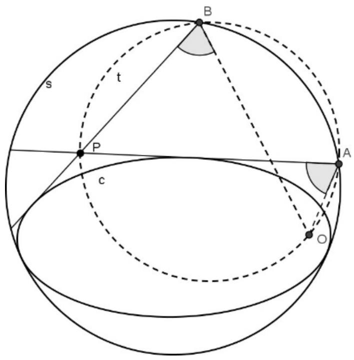

Figure 10.10: Peripheral angle statement

298. The theorem provides a generalization of the construction above because it provides a specific metric theorem:

Metric law 17. *Given a conic section circumscribed by a circle. From a point on the periphery we draw a tangent, and a line through a focal point. When we move the point, the size of the angle will be the same (Fig.10.10)*

When we study the figure, we see that the lines from A and B are both tangents to *c*, and that they går through the focal point. At the same time, two lines på meet the circle t, and here the peripheral angle says that they must be equal in size.

299. The theorem will gå over to the parabola theorem over når the conic section becomes a parabola. Then the enclosing circles will become a line. Another variant is when the second circle becomes a line. This occurs når when the two tangents to the conic section become parallel, and the circle må gå passes through a point at infinity.

Figure 114. *Given a conic section, a circle that circumscribes it, and two parallel tangents to the conic section. Then a line through two of the intersections between the tangents and the circle will also ăgă pass through the focal point.*

If the circumscribed circle has the same center as the conic section, this fact will also justify the envelope theorem.

300. The two tangents can also be tangent to the cone intersection where it is tangent to the enclosing circle. Then we will get a symmetrical situation, and a circle that goes through both focal points.

Figure 115. *Given a conic section, an enclosing circle, and a tangent at each of the tangent points between the two. A circle through the tangent points, and through the point of intersection of the tangents, will then also ăgă through the focal points of the conic section.*

From this theorem, we can solve several tasks related to conic section normals.

301. Something special also occurs if the two tangents coincide to form a tan- gent. The point between the tangents then becomes the tangent point with the conic section. The two points of intersection with the circle merge into one, but if we follow the infinitesimal process here we realize that the circle through the points of intersection becomes tangent at this point.

Figure 116. *Given a conic section, a tangent to it, and a circle that circumscribes it. A circle that is tangent to the first circle at a point of intersection between the line and the circle, and that g ar through the focal point, will also ĝă through the point where the tangent meets the conic section.*

302. The sentences we have seen f ar a particular nuance n ar we have to do with hyperbolae, and the tangents are the asymptotes. We f ar then several sentences that give properties of the hyperbola and its asymptotes.

Figure 117. *Given a hyperbola and its asymptotes, and a circle that doubles the hyperbola. A circle through the center of the hyperbola, and through two of the points of intersection with the asymptotes, will also ă g pass through a focal point of the hyperbola.*

Figure 118. *Given a hyperbola, its asymptotes, and a circle centered at a focal point tangent to the asymptotes. Another circle centered on the center of the hyperbola that passes through the tangent points will tangent the hyperbola.*

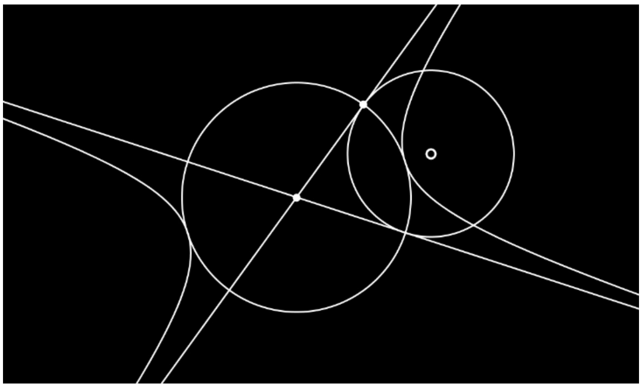

Figure 10.11: Hyperbolic circles

10.6 Imaginary tangent

303. Several relationships are revealed when the conic section and the enclosing circle are imaginatively tangent to each other. A connection is also evident in the transition between the two shapes, especially in the case where the enclosing circle tangents the conic section at four points. Then the parallel theorem above 126 first takes on a special form.

Figure 119. *A circle is quadruple tangent to a conic section. Lines through the tan miter point and through the focal points intersect the circles at two points, and the line through these points is tangent to the conic section.*

If we continue f we have imaginary tangent, and parallel sentences can n a be used to a find the focal point in the conic sections.

304. When the imaginary tangent conic section remains in the center of the enclosing circle, then this itself becomes a circle. Thus arises the following theorem:

Figure 120. *Given two concentric circles, and two tangents to the inner circle.A circle through two of the tangents' intersection points with the outer circle that g passes through the intersection of the tangents will also a g pass through the center of the circles.*

305. The theorem above shows the theorem about broken chord:

Metric law 18. *Given a circle, two points A and B p a the arc of the circle, and a third point C p a the arc connected by chords from A and B. From the point M midway between A and B, a normal is joined to F, and this will halve the distance from A to B via C.*

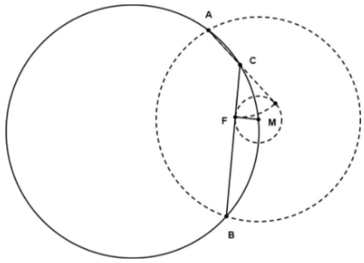

Figure 10.12: Kinked chord

We realize this from the geometric relationship above by studying the tan genes.

Chapter 11

Equilibrium

In the last family of theorems, we have an equilibrium between circle points and focal points, or between the infinitely large circle and the infinitesimally small circle, which are nevertheless connected to each other. This is already expressed in a certain number of theorems which are to some extent astonishing. One is given thus: Given a parabola and three tangents to it. A circle circumscribing the parabola will also pass through the focal point of the parabola. And we have the construction. Given a circle and a point inside it. Lines are drawn through the point, and normals are erected where the lines meet the circle. The normals then enclose an ellipse that touches the circle and has the point as its focal point. The peculiar thing about these theorems is that they do not arise by a circle becoming infinitely small, and by a circle becoming infinitely large. It seems that the focal point is formed by two imaginary lines where the two lines come from separate conic sections. And likewise with the points, each of the focal points originates from its own conic section. We see this in a classic transition from a real to an imaginary image. This transition is from an image that is also known as a special case of Poncelst's theorem. Given a conic section with two triangles inscribed. Then the triangles will also inscribe a conic section. The theorem also implies: Given a conic section that inscribes a triangle where the triangle in turn inscribes a conic section. Then there is a linear set of triangles that can be placed between the two conic sections. [1] These considerations can be continued somewhat, but we now turn our attention to some circle theorems of a slightly different nature. the theorems are in many ways similar to those we have seen, it turns out that they do not come out of the basic circle theorem, but that we must take a slightly different path.

[1] This is called Poncelet's theorem, but for all polygons; if a polygon lies between two conic sections, then there is an infinite number of conic sections between them.

11.1 Two-circle focal point sentences

In the considerations we have just made, we have seen that we can make movements from circles to focal points. The reverse movements are also °often found when °we have a relationship that involves a focal point, then we expect that this can be resolved into a double-aspect circle. Surprisingly, °this does not always turn out to be possible, and one theorem where this is not the case is a well-known theorem related to the parabola.

Figure 121. *Given a parabola and a triangle that circumscribes it. A circle that circumscribes the triangle will then g ã through the focal point of the parabola.*

This theorem cannot be generalized so that the focal point becomes a circle that is double tangent to the parabola, and we will see the reason for this in a moment.

It turns out that the theorem can be traced back to the double triangle theorem from the previous chapter.[2] Here we had the fact that n°ar two triangles are inscribed in a conic section, then they also °rewrite a conic section. We let n°a two points p°a the same triangle be the circle points. This causes the circumscribed conic section to become a circle. The line through the circle points m°a be the line at infinity, and the inner conic section that is tangent to this line m°a then be a parabola. The two tangents through the circle points form the focal point of this parabola because they are tangents to it. The other triangle is not affected by all this and it circumscribes the parabola. The circle rewrites this , and also °g o e s °through the sixth point, which is the focal point of the parabola.

Here we have °two theorems that have the same structure, but which behave completely differently w h e n °the circle points are taken into account. We realize n°å also °a the reason why the focal point cannot be a circle; this is because the two lines that form the focal point come from different conic sections. We saw°a how two opposite °sides of the hexagon came from a conic section; we must°a therefore have a line from each triangle's triangle to°a form the conic section, here two lines in the same triangle form the focal point.

This consideration leads us to°ask for the most general configuration that allows a line from each conic section to form a focal point. It turns out°to be a variant where we from the main theorem have let two conic sections become two pairs of points, and two others become two pairs of lines, so that the configuration is self-dual.

Figure 122. *Given a conic section that intersects another at four points, and a third that double tangents this. Through each of the four intersection points*

[2]The observation we are making here was made in some book that we cannot remember.

we add a tangent to the double-tangent conic section. There is then a conic section that doubles the other conic section and the four lines.

We see here that two of the conic sections have four common points, and that the other two have four common lines.

We let n°a two of the points originating from each conic section become the circle points. Both conic sections through these will then be circles, while the other two conic sections will °b e the common focal point of the two lines through the circle points. We can then set up the theorem:

Figure 123. *Given two conic sections with a common focal point, and a circle that doubles each of these. These are laid out so that a common tangent to the conic sections, g °a r through an intersection point of the circles. The other joint tangent g° will then pass through the other intersection between the circles.*

We note that this theorem is completely dual. Furthermore, we can consider that we cannot let the two lines be a conic section; we do not °have in general that there is a conic section that double tangents each of the two conic sections, and g o e s °through the intersections of the circles.

This theorem turns out to have a number of properties. We will make an initial assessment that shows a specific property, before we look at a more general variant.

We allow one of the cone sections to expand so that it eventually becomes a circle. This circle will then have the common focal point as its center. The circumscribed circle will coincide with this circle, and we have the following.

Figure 124. *Given a conic section, and a circle with the same center circumscribes this. Another circle has its center at one of the focal points of the conic section, and a tangent to this circle where it intersects the first will also °be tangent to the conic section.*

From this theorem, we °immediately see the envelope direction, because we realize that the tangents must °be perpendicular to lines from the focal point to the tan point.

We also °a come to this theorem by °a establishing another variant. We let n °a one of the conic sections become narrower and narrower, and as it collapses it becomes a line segment, of which only the endpoints are significant. We then f °a r the following conditions:

Figure 125. *Given a conic section, a circle that circumscribes it, and two tangents to it. A circle through the common point of the tangents, and through two of the*

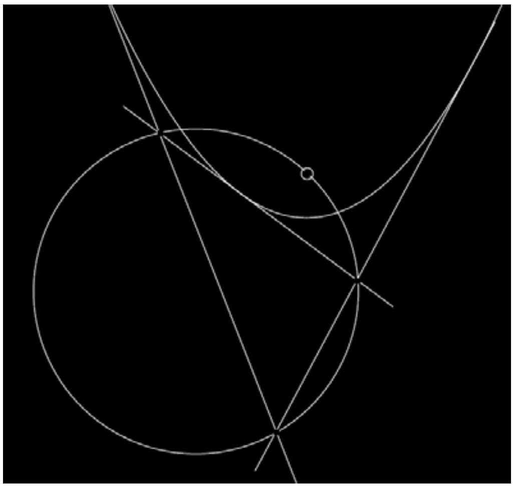

Figure 11.1: Parabola and focal point

The intersections between these and the circle will also g̊ā through the focal point the conic section.

This is a simple and striking theorem. What is ˀto note again is that ˀthe focal point in this case cannot be resolved into a circle because it is formed by two lines from separate conic sections.

This will g̊a over to the parabola statement above n ° a r the conic section becomes a parabola. Then the enclosing circles will become a line. Another variant is w h e n ˀthe second circle becomes a line. This occurs n ° w h e n the two tangents to the conic section become parallel, and the circle m̊g̊a passes through a point at infinity.

Figure 126. *Given a conic section, a circle that circumscribes it, and two parallel tangents to the conic section. Then a line through two of the intersections between the tangents and the circle will also ā̊gā pass through the focal point.*

I f ° the circumscribed circle has the same center as the conic section, this fact will also ˀjustify the envelope theorem.

The two tangents can also ˀbe tangent to the cone intersection where it is tangent to the enclosing circle. Then we will get ° a symmetrical situation, and a circle that g o e s ° through both focal points.

Figure 127. *Given a conic section, an enclosing circle, and a tangent at each of the tangent points between the two. A circle through the tangent points, and*

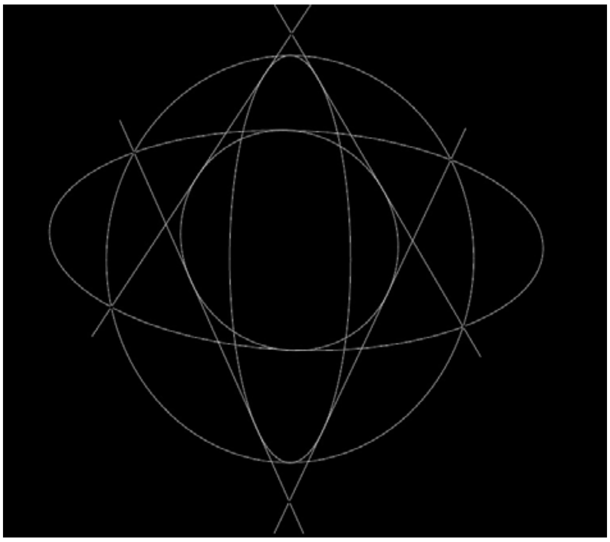

Figure 11.2: Output for the focal point circle theorem

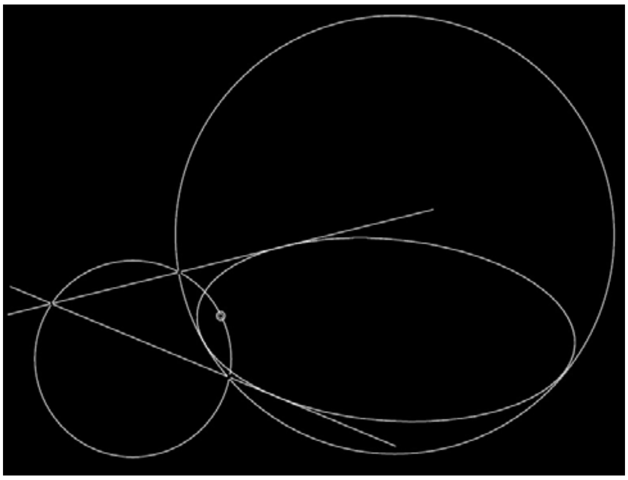

Figure 11.3: Circle through focal point

through the point of intersection of the tangents, will then also gã through the focal points of the conic section.

From this theorem, we can solve several tasks related to conic section normals.

Something special also occurs if the two tangents merge into one tangent. The point between the tangents then becomes the tangent point with the conic section. The two points of intersection with the circle merge into one, but if we follow the infinitesimal process here we realize that the circle through the points of intersection becomes tangent at this point.

Figure 128. *Given a conic section, a tangent to it, and a circle that circumscribes it. A circle that is tangent to the first circle at a point of intersection between the line and the circle, and that g ar through the focal point, will also gã through the point where the tangent meets the conic section.*

The sentences we have seen f ar a particular nuance n ar we have to do with hyperbolae, and the tangents are the asymptotes. We f ar then several sentences that give properties of the hyperbola and its asymptotes.

Figure 129. *Given a hyperbola and its asymptotes, and a circle that doubles the hyperbola. A circle through the center of the hyperbola, and through two of the points of intersection with the asymptotes, will also ã g pass through a focal point of the hyperbola.*

Figure 130. *Given a hyperbola, its asymptotes, and a circle centered at a focal point tangent to the asymptotes. Another circle centered at the center of the hyperbola that passes through the tangent points will tangent the hyperbola.*

11.2 Imaginary tangent

Several relationships are revealed when the conic section and the enclosing circle are imaginarily tangent to each other. A connection is also evident in the transition between the two shapes, especially in the case where the enclosing circle is tangent to the conic section at four points. Then the parallel theorem above 126 first takes on a special form.

Figure 131. *A circle is quadruple tangent to a conic section. Lines through the tan miter point and through the focal points intersect the circles at two points, and the line through these points is tangent to the conic section.*

If we continue f we have imaginary tangent, and parallel sentences can na be used to a find the focal point in the conic sections.

W h e n the imaginary tangent conic section remains in the center of the enclosing circle, then this itself becomes a circle. Thus arises the following theorem:

Figure 132. *Given two concentric circles, and two tangents to the inner circle. A circle through two of the tangents' intersection points with the outer circle that g passes through the intersection of the tangents will also ag pass through the center of the circles.*

This can also be further specialized so that the two keys coincide.

We have thus come to a conclusion n regarding the main considerations.

11.3 Tasks

6. ***Cracked cork***
Given a circle and three points A, C and B pa the periphery, and line segments AC and BC. From the midpoint p of the arc between A and B we drop a normal, and have the point D where it hits. Show that the distances along the lines from A to D and from B to D are the same length, even though in one case we have a broken line.

We apply the theorem above. Figure 11.4 shows that the broken chord ACD is as long as the tangent AE, which in turn is as long as BD.

7. *Given a conic section at its major axis and focal points, and a tangent to the conic section. Find the point where the tangent touches the conic section.*

8. *Given a circle, and two points equidistant from the center. Find the shortest path between the points via the periphery of the circle.*

9. *Construct the focal point of a parabola tangent to four lines.*

10. *A parabola is given by a focal point and three tangents. Find one of the parabola's tangent points with the tangents.*

11. *Construct a parabola that is tangent to four lines.*

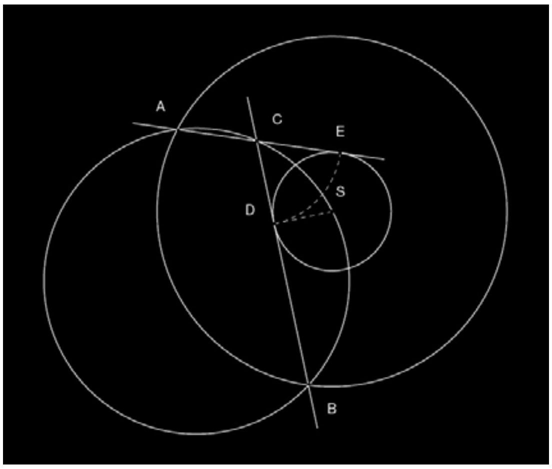

Figure 11.4: Theorem broken chord

We find the focal point by°finding the intersection of two circles that circumscribe two triangles. We can find new lines by°a adding circles that g°ar through the focal point and the intersection of two tangents. The new tangents are found between the other intersections. Points on°a the tangents are found by°a letting circles intersect tangents where others intersect .focal point

12. *A hyperbola and its asymptotes are given. Find the focal points of the hyperbola.*

Part III

Appendix

Chapter 12

Expansions

1. The basic type can be expanded, or used in conjunction with several other theorems, and we will look at some of these

12.1 Extension to the room

2. The type can immediately be extended to space. We then have to run with second degree curves in space, or quadrics as they are also called. These are the basic elements, but the basic connection is tagging along a spline. In other words a two quadrics touch each other continuously along a conic section, and this conic section will also a lie in a plane.

3. Unlike conic sections, a phenomenon already occurs at the connection between four squares.

Proposition 1. *Given a square, and two others that are tangent to it. Then there is a linear set of squares that touch these two.*

The same applies to the conic sections, but this is complicated since the conic sections have five degrees of freedom. There it will always be the case that two conic sections have a linear number of conic sections that affect them both, but this is not the case here. In general, two arbitrary squares will not touch many others, but if they have a square in common, then they will have a linear set in common.

12.2 Degeneration of squares

4. A square that is tangent to two other passages ar p a same m ate as the wedge-shaped sections has four special branches, two zero points s a and si and two poles. In the

In one case, the square becomes a plane, with a conic section as a boundary, in the other case a point, with a cone through it.

5. This also implies that two squares contained in a third also has two cones in common, and two common planes, with cone sections in.

6. In space, one can also ap a another m ate continuously follow the transition from real to imaginary tangent. W hen we cut two surfaces that are connected to each other through their contact conic sections, the image will be two conic sections that double-tangent each other. (In particular, an entire conic section image will emerge w h e n the entire constellation is cut.) If we let n a section plane rotate so that it still intersects the conic sections, but not the tangent curve, then the two conic sections will touch each other imaginatively.

7. We can make this observation with a cone and a sphere that are tangent to each other in a circle. The image of this in a plane that intersects the sphere, cone and circle will be a circle that double tangents a conic section. W h e n the plane no longer intersects the circle, but still intersects the sphere and the cone, the image will be a circle that d-tangles the conic section imaginatively. W h e n the plane is tangent to the sphere, the circle will be infinitely small, but it will still be d-tangent to the conic section, and will thus be the focal point of the conic section. This brings us back to the initial definition of the focal point based on the conic section.

8. This means that two squares tangent to a common square will intersect along two conic sections, while the two generally intersect along a fourth degree curve in space.

12.3 Imaginary circle Brennpunkt and brennsirkel

9. In the plane, the circle can be seen pa as a conic section p a s s i n g through the two imaginary circle points at infinity. W h e n we apply similar considerations in space, we find that spheres in space can be seen pa as squares that g a r through an imaginary circle at infinity.

10. Munge's theorem and the diagonal triangle theorem apply immediately here.

12.4 Double cube

11. Another generalization that is central is to expand the structure. The structure of the type is the octahedron. This number is a power of two, and can also be written as a Pascal series, i.e. it is the sum of the numbers 1, 3, 3, 1. We describe the type in this order. First a primary conic section, then three secondary ones, then three tertiary ones, and finally the final or quaternary one. This can be

We expand by starting with a conic section and allowing four to intersect it. Between the four, we find six that are tangent to two and two. Then we find four that are tangent to three and three of the previous ones, and finally one that is tangent to the four. This is a double cube structure that has 16 conic sections that all touch 4 others.

12. This also sits in space, and here we can form a clear picture of the case. We let the imaginary circle at infinity be the primary square. Through this g ar four spheres, between the four spheres we find four planes. Then three and three planes will meet in four lines, and these four lines will go through the same point.

13. Åfa such forward in the plane is not always easy because the end element often becomes imaginary. A variant where the element appears is when we start with four circles, find the six double keys between them, then the four conic sections linked to three and three. These four conic sections will then all touch a fifth conic section.

Chapter 13 Tasks

The conic section doctrine is associated with many problems, some of a complicated nature. The master in this field was Jacob Steier, who presented and solved countless problems of various kinds.

In this presentation, we will not tackle all types of problems, but will limit ourselves in two ways. One is that we shall remain within the morphological area in which we work when applying the method; that is to say, when solving problems we will use the theorems that emerge from the morphological investigations. We do not draw on theories outside this area to find a solution. In the beginning of the assignments we will still use the elementary circle constructions that a find normals, parallel lines and other things before these appear as theorems. Eventually, however, we will see that these simple constructions also emerge from the topic.

The second limitation concerns the choice of assignments. In general, we will not outline tasks out of the blue a. On the one hand, there will be simple classroom tasks that are easy to understand that we also a extend to more general images. On the other hand, the assignments will be of such a nature that we can make sense of the various images that appear.

13.1 Desargues theorem and parallelism

N f we don't have a metric, parallelism is justified in relation to given parallelism. If two parallel lines are given, we can find parallels to this one, and if two pairs of parallel lines are given, we can find parallel lines to all lines.

13. *Given two parallel lines, and a point beyond them. Find a parallel to the lines through the point.*

14. *Given two pairs of parallel lines, and a single line that is not parallel to these. Find a parallel to this line.*

15. *Given two pairs of parallel lines, a single line that is not parallel to these and a point that does not lie on either line. Find a parallel to the single line through the point.*

Another aspect of Desargues' theorem is that each point can be a perspective point. Each new point we choose results in two new triangles, and a new Desargues line.

16. *Given a Descartes configuration. Starting from another point as the perspective point, find the perspective triangles, and the Desargues line. This can be repeated with other points.*

13.2 Pascal's theorem

Using Pascal's theorem, we can construct a number of basic elements. The first is that once we have given five points to a conic section, we can find as many new ones as we like.

17. *Given five points on a conic section. Find a new point on the conic section.*

18. *Given a conic section at five points, and a line through one of the points. Find the second intersection of this line with the conic section.*

19. *Given a conic section through five points, and a line through two of these. Through one of the other intersections we draw a parallel to the line. Find the line's second point of intersection with the conic section.*

20. *Given a conic section at five points. Find a diameter for the conic section.*

21. *Given a conic section at five points. Find the center of the conic section.*

22. *Given a conic section at five points. Find a tangent at one of the points.*

23. *Given a pascal configuration with given point order (1,2,3,4,5,6) and find the pascal line. Then cyclically permute the first three points so that we find the point orders (3,1,2,4,5,6) and (2,3,1,4,5,6). Find the Pascal lines also here. What is the result?*

13.3 loo

The first type of construction we have to deal with is the formation of curves. From Pascal's theorem we have that we can find as many points as we want when five are given. When this is in place, we can consider to have found a conic section when we have a point locus.

13.4 Brianchon's theorem

By Brianchon's theorem, we can construct the dual images of Pascal. However, the parabola is special here because it is tangent to the line at infinity, which provides more tasks.

24. *Given four tangents to a parabola. Find a fifth tangent.*

25. *Given four tangents to a parabola. Find a line parallel to the axis.*

26. *Given a conic section and a line outside this. Find a diameter parallel to the conic section.*

27. *Given a conic section at five points. Find a diameter of the conic section parallel to a given line.*

28. *Given a hyperbola by the two asymptotes and a line. Find the point of tangency of the hyperbola with the line. Where is the point in relation to the line's intersection with the asymptotes?*

29. *A hyperbola is given by the two asymptotes and a line. Find another tangent to the hyperbola.*

13.5 Perseverance

Here we look at a construction method.

30. *Given a circle, and a diameter. An ellipse has the same axis, and we have given a point p to this. Find another point på the ellipse.*

31. *Given a circle, and a diameter. An ellipse has the same axis, and it is tangent to a given line. Find the tangent point between the line and the ellipse.*

32. *Given two conic sections that intersect at two points. Find a common point for the two.*

33. *Given two conic sections that intersect at four points. Construct their four joint tangents.*

34. *Given two conic sections that meet at two points. Construct their inner trap point.*

35. *Given two conic sections in perspective that do not intersect. Construct their perspective lines.*

13.6 Tangency and cutting

In many of the conic section geometry tasks, it is a point to be able to construct intersection points and tangents to conic sections when five points or five lines are given. In a geometry program, we can find the conic section through the five points, and then find the intersection points. Here, however, we will see what is possible in principle with a circle and ruler. Once this has been done, you can use the possibilities offered by the various tools.

36. *Given a conic section through five points, and a line through two of these. Through one of the other intersections we draw a parallel to the line. Find the line's second point of intersection with the conic section.*

37. *Given three points på conic section and a circle through the three points. Find the fourth intersection between the circle and the conic section.*

38. *Given a conic section at five points and a line. Find the intersection points between the conic section and the line.*

39. *Given a conic section at five points, and a circle through two of the points. Find the other intersections between the circle and the conic section.*

13.7 Appolonisu constructions

One class of constructions are the såklate appolonuius constructions. Here, the task is to construct circles that tanerer three given circles. Nar the circles can assume their extreme values.

40. *Construct the diagonal between two circles that do not intersect.*

41. *One circle is inside another. Construct the diagonal between them.*

42. *Construct the outer joint bars into two circles.*

43. *One circle is inside another. Construct the diagonal between them.*

44. *Construct the diagonal between two circles that do not .*

45. *One circle lies inside another. Construct the diagonal between them.*

46. *Construct the common points between two circles that do not .*

47. *Construct the outer joint bars into two circles.*

48. *Find the outer common line of two circles.*

49. *Given two circles and a point on one of the circles. Construct a circle that is tangent to one of the circles in the point, and that is also a tangent to the other.*

50. *Construct one of the inner common lines of two circles.*

51. *Construct a circle that passes through three given points.*

52. *Construct one of the circles tangent to three given lines.*

53. *Construct one of the circles that passes through two given points and is tangent to a given line.*

54. *Construct one of the circles tangent to two lines and går through a given point.*

55. *Given a circle and two points. Find the circles that go through the points and are tangent to the circle.*

56. *Given a circle and two lines. Find the circles that are tangent to the lines and the circle.*

13.8 General constructions

Here, we extend the Appolonius construct to apply in general.

57. *A conic section g passes through four points and is tangent to a straight line. Find the point of tangency with the line.*

58. *A conic section is tangent to four lines and passes through a given point. Find the tangent at the point.*

59. *A conic section passes through three points and is tangent to two lines. Find the tangent points with the lines.*

60. *A conic section is tangent to three lines and passes through two points. Find the tan genes in the points.*

61. *Given a conic section and two points. A conic section g passes through two given points på the conic section, through the two points, and is tangent to the conic section. Find the tangent point.*

62. *Given a conic section and two lines. A conic section is tangent to the lines, tangent to the conic section, and passes through two points on the periphery. Find the tangent point between the two conic sections.*

63. *Given a conic section, two tangents to this, and two lines. Another conic section is tangent to the four lines and the conic section. Find the tangent point between the two.*

13.9 focal point

A conic section with a given focal point is dual to a circle. We can therefore find the elementary circle constructions here, and also the Appollonius constructions. In these exercises, we can imagine the conic sections as given, so that we can find the intersection points between these and a line immediately.

64. *Given a conic section. Find the focal point of this.*

65. *Find the directrix of a given conic section with a focal point.*

66. *Given a conic section at its one focal point, the directrix, and one more point p at the periphery. Find another point p at the periphery.*

67. *Given two conic sections with a common focal point that do not . Find the diagonals between them.*

68. *Given two conic sections with a common focal point. Find the common point between them.*

69. *Given a conic section at its focal point and three points on its periphery. Find the directrix of the conic section.*

70. *Given a focal point and three lines. Find the conic section with this focal point tangent to the lines.*

13.10 Duality

71. ***Tangents from point to conic section***
Given a conic section, and a point outside the conic section. Find the tangents from the point to the conic section.

We draw two lines from the point above the conic section, and find lines through the intersection points. These meet at two points on the pole, and where the pole intersects the conic section we have the tangent points. We can now draw the tangents.

72. *Show the theorem:*
Given a hyperbola and its asymptotes, and two tangents to the hyperbola. The intersection of the tangents, the center of the hyperbola, and the intersection of two parallels with the asymptotes through intersections with them will be
p has the same line.

73. ***Cracked cork***
Given a circle and three points A, C and B p a the periphery, and line segments AC and BC. From the midpoint p of the arc between A and B we drop a normal, and have the point D where it hits. Show that the distances along the lines from A to D and from B to D are the same length, even though in one case we have a broken line.

74. *Given a conic section at its major axis and focal points, and a tangent to the conic section. Find the point where the tangent touches the conic section.*

75. *Given a circle, and two points equidistant from the center. Find the shortest path between the points via the periphery of the circle.*

76. *Construct the focal point of a parabola tangent to four lines.*

77. *A parabola is given by a focal point and three tangents. Find one of the parabola's tangent points with the tangents.*

78. *Construct a parabola that is tangent to four lines.*

We find the focal point by°a finding the intersection of two circles that circumscribe two triangles. We can find new lines by°a adding circles that g°ar through the focal point and the intersection of two tangents. The new tangents are found between the other intersections. Points on°a the tangents are found by°a letting circles intersect tangents where others intersect .focal point

79. *Construct a circle tangent to a given conic section and a given line.*

80. *Construct a circle tangent to a given conic section and a g°ar through a given point inside the conic section.*

81. *Given a conic section and a point pån the axis. Find the normal from the point påf the conic section.*

13.11 To brennpunkt

82. *Given a conic section at two focal points and a point påt the periphery. Find by construction a new point påt the periphery.*

83. *Given two circles. A parabola is tangent to both of these. Construct a point påt the periphery of the parabola.*

13.12 firing compass

84. *Given a conic section and a point outside it. Find a circle that encloses the conic section and g°ar through the point.*

Given a conic section and the vertical axis and a point på this. Find the normals from the point to the conic section.

85. *A hyperbola and its asymptotes are given. Find the focal points of the hyperbola.*

86. *Given a conic section at two focal points and a line. Find the tangent point with the line.*

87. *A conic section is given by a focal point and three tangents. Find the conic section at to find the second focal point.*

88. *Given a conic section by a circle, a focal point and a line. Find the tangent point between the tangent and the conic section.*

89. *Given a conic section by a circle and a focal point and a point in addition. Find the tangents in the point. How many solutions do we have?*

90. *Given a circle and two points inside the circle that are equidistant from the periphery. Find the fastest path from one point to the other via the periphery.*

91. *Given a parabola, the axis and a point på this. Find the normals from the point på the parabola.*

92. *Given a conic section by a circle and three lines. Find a focal point for the conic section.*

13.13 Proposed solutions

(22)We draw a five-pointed star between the points, and where one diagonal intersects the line between two intersection points, we draw the line to the fifth point. This is a tangent.

(18) We use Pascal's theorem and let the searched point be the sixth point in a Pascal configuration. We find the Pascal line by two pairs of lines, and the third Pascal point is found where this intersects the given line. When we have this point, we can draw the last line in the Pascal hexagon, and we find the searched point.

Chapter 14

Images

Here is a systematic review of the images.

14.1 The starting point

1. **Desargues theorem**
Given two triangles in perspective. Then matching sides of the triangle will meet in three points that all lie on line. We call this line the Desargues line, and we can call the perspective point the Desargues point in the configuration.

2. **Desargues' theorem** N a r two triangles are point perspective, then they are also a line perspective.

3. Given two triangles with vertices p a three parallel lines. Then two and two matching lines in the triangle will meet at three points p a the same line.

4. If two pairs of lines in a hexagon inscribed in a circle are parallel, then the third pair will also be parallel.

5. **Pascla's theorem** Given a hexagon inscribed in a conic section. Then opposite sides of the hexagon will meet at three points that all lie on the same line.

6. Given two lines, and three points on each of the lines. We form intersections of lines between between two and two points on each line, and the intersection points are on the same line.

7. Given a hexagon inscribed in a conic section, where two opposite sides are parallel. Then the Pascal line of will be parallel to these lines.

8. If two pairs of opposite sides of a hexagon inscribed in a conic section are parallel, then the third pair of opposite sides will also be parallel.

9. Given a pentagon inscribed in a conic section. Then we want a tangent at a point, a line in the pentagon, and a line through two intersections of the other four lines, go a through the same point.

10. **McLaren's statement**
Given a square inscribed in a conic section. Then opposite sides, and opposite tangents will meet at points that all lie on the same line.

11. **McLaren's second theoremGiven** a square inscribed in a conic section, and tangents at two neighboring points in the square. Sides of the square and tangents meet at two points, the line between the points of tangency and the opposite side of a point, and these points lie on the same line.

12. Given a triangle inscribed in a conic section. Then one side of the triangle will meet the tangents in opposite corners at three points that lie on the same line.

13. Given a hyperbola and its asymptotes, and two points on its periphery on either side of the center. Parallels with the asymptotes are drawn through the points, and the line through their intersections will also go pass through the center.

14. Given a hyperbola and its asymptotes (a and b), and two points (P and Q) on the periphery. A line through P parallel to one asymptote, and a line through Q parallel to the other, intersects these at two points, and the line through these is then parallel to the line PQ.

15. Given an ellipse, two tangents to it, and two points on the periphery such that the line through them is parallel to the line through the tangent points. We draw lines between the tangent points and points on the periphery, and where these meet the tangent points are formed, and lines through these are then parallel to the line through the tangent points.

16. Given a conic section, and two parallel tangents to it. From two points on the periphery of the conic section, we draw lines through the tangent points, and these will intersect at two more points. The line through these points is then parallel to the tangents.

17. Given a parabola, a tangent to it, and two points on the periphery. From the points on the periphery, we draw lines to the tangent point, and lines parallel to the parabola. These meet at two points, and the line through these points is parallel to the tangent.

14.2 Duality

18. **Brianchon's theorem**
Given a conic section and a hexagon that circumscribes this. The three main iagons

The links in the hexagon will then meet at the same point.

Brianchon's theorem appears, if possible, even simpler in expression than Pascal's theorem, and many transformation possibilities are also associated with it, and we will look at a few of these before we later look at the duality principle from another angle.

19. Given a conic section circumscribed by a pentagon. We find two diagonals in the pentagon. A line through the fifth corner of the pentagon that passes through the opposite tangent point will also pass through the intersection point between the diagonals.

20. **Dual Mclaren**

Given a conic section, and a square that circumscribes this. A line between two of the tangent points will then go through the point where the diagonals of the square meet.

21. Given a conic section m, and a square that circumscribes this. Two lines between tag ring points and corners in the square will meet on a diagonal in the square.

22. Given a triangle, and a conic inscribed in it. The lines between the corners of the triangle and the tangent points will meet at the same point.

23. Given a parabola, two tangents to it, and the line between the tangent points. We find parallels to the tangents that meet on the transversal, and where these meet the tangents we draw a line. This is also tangent to the parabola.

24. Given a hyperbola, its asymptotes and two tangents. The lines through the points where the tangents meet the asymptotes are then parallel.

25. Given a parabola, three tangents to it, and a diameter of the parabola through one of the tangent points between the parabola and one of the tangents. Where this intersects the other lines, we draw parallels to the third, and these will meet at the diameter.

26. Given a pole and a polar. We let a line go through the point. The pole of this line then lies on the polar of the line.

27. Given a conic section, a point, the two tangents from the point to the conic section and the polar. We add a point to the polar, subtract tangents from this, and find the polar. This will then go through the original point.

28. Given a conic section, a point outside the conic section, the two tangents, and the polar of the point. We draw two lines through the pole, and these intersect the conic section at four points. Lines through these points will then meet at the pole.

29. Given a conic section, a line cutting across this, the two tangents and the pole of the line. At the pole we add two points, and from these four tangents to the conic section. The tangents form common points, and lines through these will go through the pole of the line.

Register